

On Certain Difference Sequence Spaces

Vats Pooja

Department of Mathematics

K.R. Mangalam University, Gurgaon, NCR Delhi, India

Abstract

The difference sequence spaces, and were introduced by Kizmaz (Can. Math. Bull. 24:169-176, 1981). In this paper, I introduce the sequence spaces and first difference sequence spaces.

Keywords: Sequence space, Banach space, Normed Linear space and subspace

I. INTRODUCTION

From the title of this write up it is obvious that it deals with some concept of difference sequence spaces. In this paper I introduce the spaces $l_{\infty, c}$ and c_0 . These are the spaces of complete bounded, convergent and null sequence respectively. I also give idea about first difference sequence spaces.

II. DEFINITIONS AND NOTATIONS

A. The spaces $l_{\infty, c}$ and c_0 : The spaces $l_{\infty, c}$ and c_0 are the linear spaces of complete bounded, convergent and null sequence respectively.

i.e. $l_{\infty} = \{x = (x_k) : \sup_k |x_k| < \infty\}$, $c = \{x = (x_k) : \lim_k x_k \text{ exists}\}$, $c_0 = \{x = (x_k) : \lim_k x_k = 0\}$

B. Normed linear space: Let X be a linear space over the field K (R or C). A function $\| \cdot \| : X \rightarrow R$ is said to be norm on X if it satisfies the following conditions:-

- $\|x\| \geq 0 \forall x \in X$
- $\|x\|=0 \Leftrightarrow x = 0$
- $\|x + y\| \leq \|x\| + \|y\| \forall x, y \in X$
- $\|\alpha x\| = |\alpha| \|x\| \forall x \in X, \alpha \in K$

C. Banach space: A complete normed linear space over field K is called a Banach space.

D. Let $(X, \| \cdot \|)$ be a normed space.

- A sequence $\{x_n\} \subset X$ is said to be convergent if \exists an $x \in X$ s. t. $\lim_{n \rightarrow \infty} \|x_n - x\| = 0$
- A sequence $\{x_n\} \subset X$ is said to be Cauchy sequence if for a given $\varepsilon > 0$, \exists a + ve integer N s. t. $\|x_m - x_n\| < \varepsilon$ $\forall m, n \geq N$
- The space X is complete if every Cauchy sequence in X converges to an element in X .

E. Subspace: A non-empty subset Y of a linear space X over K is a subspace of X if and only if $x+y \in Y$ and $\alpha x \in Y, \forall x, y \in Y$ and $\alpha \in K$

F. Closed subspace: A subspace Y of a normed space X is called a closed subspace of X if Y is closed in X considered as a metric space.

III. FIRST DIFFERENCE SEQUENCE SPACES

The study of difference sequence spaces was initiated by Kizmaz [chand.Math.Bull.24(1981)], with the introduction of the following sequence space.

$$l_{\infty}(\Delta) = \{x = (x_k) : \Delta x \in l_{\infty}\}, C(\Delta) = \{x = (x_k) : \Delta x \in c\}, c_0(\Delta) = \{x = (x_k) : \Delta x \in c_0\}$$

Where $\Delta x = (\Delta x_k) = (x_k - x_{k-1})$. It was shown that these spaces are Banach spaces with the norm $\|x\|_{\Delta} = |x_1| + \|x\|_{\infty}$.

A. Theorem:

$l_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$ are the Banach spaces with the norm defined by $\|x\|_\Delta = |x_1| + \|\Delta x\|_\infty$.

B. Proof:

We prove the theorem for $l_\infty(\Delta)$. Other spaces can be treated similarly. We can easily show that $l_\infty(\Delta)$ is a normed linear space. Clearly $\|x\|_\Delta \geq 0 \forall x \in X$, Suppose that $\|x\|_\Delta = 0$

- $\Rightarrow |x_1| + \|\Delta x\|_\infty = 0$
- $\Rightarrow |x_1| + \sup_k |x_k - x_{k+1}| = 0$
- $\Rightarrow |x_1| = 0, \sup_k |x_k - x_{k+1}| = 0$
- $\Rightarrow x_1 = 0, |x_k - x_{k+1}| = 0$ for each $k \in \mathbb{N}$
- $\Rightarrow x_k = 0$ for each $k \in \mathbb{N}$
- $\Rightarrow x = x_k = 0$ (The Zero Sequence)

Conversely,

If $x=0$, then $\|x\|_\Delta = |x_1| + \sup_k |x_k - x_{k+1}| = 0+0=0$

**C. Let $x, y \in l_\infty(\Delta)$ Then $\|x + y\|_\Delta = |x_1 + y_1| + \sup_k |(x_k + y_k) - (x_{k+1} + y_{k+1})|$
 $\leq |x_1| + |y_1| + \sup_k (|x_k - x_{k+1}| + |y_k - y_{k+1}|)$
 $= |x_1| + \sup_k (|x_k - x_{k+1}|) + |y_1| + \sup_k (|y_k - y_{k+1}|)$
 $= \|x\|_\Delta + \|y\|_\Delta$**

D. For any scalar α , we have $\|\alpha x\|_\Delta = \|\alpha x_1, \alpha x_2, \alpha x_3, \dots\|_\Delta$

$$\begin{aligned} &= |\alpha x_1| + \sup_k |\alpha x_k - \alpha x_{k+1}| \\ &= |\alpha| |x_1| + |\alpha| \sup_k (|x_k - x_{k+1}|) \\ &= |\alpha| \{ |x_1| + \sup_k (|x_k - x_{k+1}|) \} \\ &= |\alpha| \|x\|_\Delta \end{aligned}$$

Thus $l_\infty(\Delta)$ is a normed space.

Next we show that $l_\infty(\Delta)$ is complete. For this, Let (x^n) be a Cauchy sequence in $l_\infty(\Delta)$ where $x^n = (x_1^n, x_2^n, x_3^n, \dots) \in l_\infty(\Delta)$ for each $n \in \mathbb{N}$

i.e. $(x_k^n - x_k^m) \in l_\infty$ for each $n, m \in \mathbb{N}$, i.e. $\sup_k (x_k^n - x_k^m) < \infty$ for each $n, m \in \mathbb{N}$

$$\begin{aligned} \text{Now } \|x^n - x^m\|_\Delta &= \|(x_1^n, x_2^n, x_3^n, \dots) - (x_1^m, x_2^m, x_3^m, \dots)\|_\Delta \\ &= \|(x_1^n - x_1^m, x_2^n - x_2^m, x_3^n - x_3^m, \dots)\|_\Delta \\ &= |x_1^n - x_1^m| + \sup_k |(x_k^n - x_k^m) - (x_{k+1}^n - x_{k+1}^m)| \end{aligned}$$

As (x^n) is a Cauchy sequence in $l_\infty(\Delta)$, so for a given $\varepsilon > 0$, \exists a positive integer N s.t.

$$\begin{aligned} \|x^n - x^m\|_\Delta &< \varepsilon \forall n, m \geq N \\ &\Rightarrow |x_1^n - x_1^m| + \sup_k |(x_k^n - x_k^m) - (x_{k+1}^n - x_{k+1}^m)| < \varepsilon \forall n, m \geq N \\ &\Rightarrow |x_1^n - x_1^m| < \varepsilon \text{ and } \sup_k |(x_k^n - x_k^m) - (x_{k+1}^n - x_{k+1}^m)| < \varepsilon \forall n, m \geq N \\ &\Rightarrow |x_1^n - x_1^m| < \varepsilon \text{ and } |(x_k^n - x_k^m) - (x_{k+1}^n - x_{k+1}^m)| < \varepsilon \forall n, m \geq N \end{aligned}$$

And for each $k \in \mathbb{N}$

$$\begin{aligned} \text{Now } |x_2^n - x_2^m| &= |(x_2^n - x_2^m) - (x_1^n - x_1^m) + (x_1^n - x_1^m)| \\ &\leq |(x_2^n - x_2^m) - (x_2^n - x_2^m)| + |x_1^n - x_1^m| < \varepsilon + \varepsilon = 2\varepsilon \forall n, m \geq N \end{aligned}$$

Similarly $|x_3^n - x_3^m| < \varepsilon \forall n, m \geq N$, Continuing like this $|x_k^n - x_k^m| < \varepsilon \forall n, m \geq N$ and $k \in \mathbb{N}$

Hence $(x_k^n) = (x_k^1, x_k^2, x_k^3, x_k^4, \dots)$ is a Cauchy sequence in \mathbb{C} . By the completeness of \mathbb{C} , (x_k^n) converges to x_k say, i.e.

$$\begin{aligned} \lim_{n \rightarrow \infty} x_k^n &= x_k \text{ for each } k \in \mathbb{N}, \text{ And } \lim_n |(x_{k+1}^n - x_{k+1}^m) - (x_k^n - x_k^m)| \\ &= |(x_{k+1}^n - x_{k+1}^m) - (x_k^n - x_k^m)| \leq \varepsilon \forall n \geq N \text{ and for each } k \in \mathbb{N} \\ &\Rightarrow \sup_k |(x_{k+1}^n - x_{k+1}^m) - (x_k^n - x_k^m)| \leq \varepsilon \forall n \geq N \end{aligned}$$

Consequently, we have

$$\begin{aligned} \|x^n - x\|_\Delta &= |x_1^n - x_1| + \sup_k |(x_{k+1}^n - x_{k+1}) - (x_k^n - x_k)| \leq \varepsilon + \varepsilon \forall n \geq N \\ &\Rightarrow \|x^n - x\|_\Delta \leq 2\varepsilon \forall n \geq N \\ &\Rightarrow x^n \rightarrow x \text{ as } n \rightarrow \infty \end{aligned}$$

Now we must show that $x \in l_\infty(\Delta)$.

For this consider

$$\begin{aligned} |x_k - x_{k+1}| &= |x_k - x_k^N + x_k^N - x_{k+1}^N + x_{k+1}^N - x_{k+1}| \\ &= |x_k^N - x_{k+1}^N - (x_k^N - x_k - x_{k+1}^N + x_{k+1})| \\ &\leq |x_k^N - x_{k+1}^N| + \sup_k |(x_k^N - x_k) - (x_{k+1}^N - x_{k+1})| \\ &\leq |x_k^N - x_{k+1}^N| + |x_1^N - x_1| + \sup_k |(x_k^N - x_k) - (x_{k+1}^N - x_{k+1})| \end{aligned}$$

$$= |x_k^N - x_{k+1}^N| + \|x^N - x\|_\Delta$$
 Science $x^N \in l_\infty(\Delta)$,
 We have $\sup_k |x_k^N - x_{k+1}^N| < \infty$ and so $|x_k - x_{k+1}| \leq |x_k^N - x_{k+1}^N| + 2\varepsilon$
 $\leq \sup_k |x_k^N - x_{k+1}^N| + 2\varepsilon = p$ (say) for each $k \in N$
 $\Rightarrow \sup_k |x_k - x_{k+1}| < \infty$
 $\Rightarrow \Delta x \in l_\infty$
 $\Rightarrow x \in l_\infty(\Delta)$
 Hence $l_\infty(\Delta)$ is a Banach space.

IV. CONCLUSION

Hence I conclude that $l_\infty(\Delta)$ is a Banach space. Similarly I can prove this for other spaces.

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