

HOPF Bifurcation in a Chemical Model

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Abstract

In this paper we have investigated the stability nature of Hopf bifurcation in a two dimensional nonlinear differential equation, popularly known as the Brusselator model. The Brusselator model exhibits supercritical Hopf bifurcation for certain parameter values which marks the stability of limit cycles created in Hopf bifurcations. We have used the Center manifold theorem and the technique of Normal forms in our investigation.

Keywords: Hopf Bifurcation, Supercritical Hopf Bifurcation, Subcritical Hopf Bifurcation, Centre Manifold, Normal Form

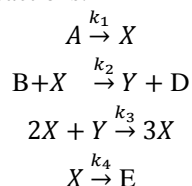
I. INTRODUCTION

The Brusselator model is a famous model of chemical reactions with oscillations. The dynamics and chemistry of oscillating reactions [27, 22, 40, 48] has been the subject of study for the last almost 60 years, starting with the work of Boris Belousov. When Belousov was studying the Krebs's cycle, he stumbled upon an oscillating system. He witnessed a mixture of citric acid, bromate and cerium catalyst in a sulphuric acid solution undergoing periodic colour changes. These changes indicated the cycle formation and depletion of differently oxidized cerium species [50]. This was the first reaction, where biochemical oscillations was observed in 1950. At this time the view point, that entropy has to increase continuously by the second law of thermodynamics until entropy reaches its maximum was so dominant that the observations which Belousov made were thought to be an artefact. Thus, Belousov couldn't manage to get his work published, till 1959 when it came out in the obscure proceedings of a Russian medical meeting. In 1961, Zhabotinsky reproduced Belousov's work and showed further oscillating reactions. Only in 1968, the results were shown to the western scientific community at a conference in Prague. The view point that entropy has to increase monotonically until a maximum is reached was shown to be restricted to closed systems. Open systems, instead, can give rise to coherent behaviour such as macroscopic flows (Bénard instability), space-time patterns and oscillations. The first theoretical explanation of oscillating chemical reactions [22, 27] was accomplished by Ilya Prigogine and G. Nicolis [18] is called the Brusselator.

The reaction mechanism, nowadays commonly called as the Brusselator, is an example of an autocatalytic, oscillating chemical reaction. Autocatalytic reactions are chemical reactions in which at least one of the reactants is also a product and vice versa. The rate equations for autocatalytic reactions are fundamentally nonlinear.

Belousov and Zhabotinsky observed that Cerium (III) and Cerium (IV) were the cycling species: in a mix of potassium bromate, cerium (IV) sulphate and citric acid in dilute sulphuric acid, the ratio of concentration of the Ce(IV) and Ce(III) ions oscillated. While for most chemical reactions, a state of homogeneity and equilibrium is quickly reached, the Belousov-Zhabotinsky reaction is a remarkable chemical reaction that maintains a prolonged state of non-equilibrium leading to temporal oscillations and spatial pattern formation.

The equations were derived from the following set of reactions:



where A and B are reactants (substrates), D and E are products and X and Y are the autocatalytic reactants of the set of reactions. Also, k_1 , k_2 , k_3 and k_4 are the rate of reactions for each component reaction.

The governing equations of the Brusselator are obtained from the autocatalytic reaction using law of mass action (i.e. the rate of a chemical reaction is directly proportional to the product of the concentration of reactant) and the set of equations for the change in concentrations of X and Y are found to be

$$\frac{d[X]}{dt} = k_1[A] + k_2[X]^2[Y] - k_3[B][X] - k_4[X]$$

$$\frac{d[Y]}{dt} = -k_2[X]^2[Y] + k_3[B][X]$$

Note that in above equations square brackets are used to denote concentrations of the reactants involved. We then nondimensionalize the equations and require the following stipulations:

$$k_3 = k_4 \text{ and } \frac{k_1 k_3}{k_2} = \frac{(k_3)^3}{(k_2)^2} = 1$$

Dropping the brackets for convenience, we then consider the free Brusselator equations given by

$$\begin{aligned}\dot{x} &= a - (1 + b)x + yx^2 \\ \dot{y} &= bx - yx^2\end{aligned}$$

where $x, y \in R$ and $a, b \in R$ and $a, b > 0$ and x and y represent the dimensionless concentrations of two of the reactants.

The term Hopf bifurcation (also sometimes called Poincare Andronov-Hopf bifurcation) refers to the local birth or death of a periodic solution known as limit cycles from equilibrium as a parameter crosses a critical value. In other words, a limit cycle is a closed solution curve for which there exist solutions that approach it asymptotically in either forward or backward time. This sort of bifurcation wherein a limit cycle is formed around an equilibrium point is called a Hopf Bifurcation. One of the tools for detecting limit cycles in the plane is the Poincare-Bendixson theorem.

Hopf bifurcation has played a pivotal role in the development of the theory of dynamical systems in different dimensions. Following Hopf's original work [1942], Hopf and generalized Hopf bifurcations have been extensively studied by many researchers [6, 11, 13, 14, 17, 20, 26, 30, 31, 37, 48, 50, 53].

In a differential equation a Hopf bifurcation typically occurs when a complex conjugate pair of eigenvalues of the linearized flow at a fixed point becomes purely imaginary. The uniqueness of such bifurcations lies in two aspects: unlike other common types of bifurcations (viz., pitchfork, saddle-node or transcritical) Hopf bifurcation cannot occur in one dimension. The minimum dimensionality has to be two. The other aspect is that Hopf bifurcation deals with birth or death of a limit cycle as and when it emanates from or shrinks onto a fixed point, the focus. Recently, Hopf bifurcations of some famous chaotic systems have been investigated and it is becoming one of the most active topics in the field of chaotic systems.

As is well known, Hopf bifurcation gives rise to limit cycles, which represent typical oscillatory behaviors of many nonlinear systems in physical, social, economic, biological, and chemical fields. These oscillatory behaviors can be beneficial in practical applications, such as in mixing, monitoring, and fault diagnosis in electro-mechanical systems [16, 26, 56]. Also, the properties of limit cycles are very useful in modern control engineering, such as auto tuning of PID controller [10] and process identification [47]. Early efforts in Hopf bifurcation control [49] focused on delaying the onset of this bifurcation [15] or stabilizing an existing bifurcation [38].

Hopf bifurcation is local, because they can be detected if any small neighbourhood of the equilibrium is chosen.

II. CENTRE MANIFOLD THEOREM AND ITS ROLE IN HOPF BIFURCATION

The center manifold theorem in finite dimensions can be traced to the work of Pliss [34], Sositaivili [5] and Kelley [4]. Additional valuable references are Guckenheimer and Holmes [26], Hassard, Kazarinoff [9], and Wan [8], Marsden and McCracken [30], Carr [24], Henry [12], Sijbrand [28], Wiggins [53] and Perko [33].

The center manifold theorem is a model reduction technique for determining the local asymptotic stability of an equilibrium of a dynamical system when its linear part is not hyperbolic. The overall system is asymptotically stable if and only if the Center manifold dynamics is asymptotically stable. This allows for a substantial reduction in the dimension of the system whose asymptotic stability must be checked. In fact, the center manifold theorem is used to reduce the system from N dimensions to 2 dimensions [8]. Moreover, the Center manifold and its dynamics need not be computed exactly; frequently, a low degree approximation is sufficient to determine its stability [7].

We consider a nonlinear system

$$\dot{x} = f(x), x \in R^n \quad (1)$$

Suppose that the system (1) has an equilibrium point x_0 at the parameter value $\mu = \mu_0$ such that $f(x_0) = 0$. In order to study the behaviour of the system near x_0 we first linearise the system (1) at x_0 . The linearised system is

$$\dot{x} = Ax, x_0 \in R^n \quad (2)$$

Where $A = Df(x_0)$ is the Jacobian matrix of f of order $n \times n$. The system has invariant subspaces E^s, E^u, E^c , corresponding to the span of the generalised eigenvectors, which in turn correspond to eigenvalues having negative real part, positive real part and zero real part respectively. The subspaces are so named because orbits starting in E^s decay to zero as t tends to ∞ , orbits starting in E^u become unbounded as t tends to ∞ and orbits starting in E^c neither grow nor decay exponentially as t tends to ∞ . Theoretically, it is already established that if we suppose $E^u = \phi$, then any orbit will rapidly decay to E^c . Thus, if we are interested in long term behaviour (i.e. stability) we need only to investigate the system restricted to E^c .

The Hartman-Grobman Theorem [14] says that in a neighbourhood of a hyperbolic critical point x_0 , the nonlinear system (1) is topologically conjugate to the linear system (2), in a neighbourhood of the origin. The Hartman-Grobman theorem therefore

completely solves the problem of determining the stability and qualitative behaviour in a neighbourhood of a hyperbolic critical point.

In case of non-hyperbolic critical point, the Hartman-Grobman Theorem is not applicable and its role is played by the centre manifold theorem. The center manifold theorem shows that the qualitative behaviour in a neighbourhood of a non-hyperbolic critical point x_0 of the nonlinear system (1) with $x \in R^n$ is determined by its behaviour on the center manifold near x_0 . Since the center manifold is generally of smaller dimension than the system (1), this simplifies the problem of determining the stability and qualitative behaviour of the flow near a non-hyperbolic critical point of (1).

III. CENTER MANIFOLD THEOREM FOR FLOWS

The statement of the Center Manifold Theorem for Flows is as follows:

Let f be a C^r vector field on R^n vanishing at the origin ($f(0) = 0$) and let $A = Df(0)$. Divide the spectrum of A into three parts, $\sigma_s, \sigma_c, \sigma_u$ with

$$\operatorname{Re} \lambda \begin{cases} < 0 & \text{if } \lambda \in \sigma_s \\ = 0 & \text{if } \lambda \in \sigma_c \\ > 0 & \text{if } \lambda \in \sigma_u \end{cases}$$

Let the (generalised) eigenspaces of σ_s, σ_c and σ_u be E^s, E^c and E^u , respectively. Then there exist C^r stable and unstable invariant manifolds W^s and W^u tangent to E^s and E^u at 0 and a C^{r-1} center manifold W^c tangent to E^c at 0. The manifolds W^s, W^u, W^c are all invariant for the flow of f . The stable and unstable manifolds are unique, but W^c need not be [24, 26].

As shown in [26, 53], the system (1) can be written in diagonal form

$$\begin{aligned} \dot{x} &= A^c x + f_1(x, y, z) \\ \dot{y} &= A^s y + f_2(x, y, z) \\ \dot{z} &= A^u z + f_3(x, y, z) \end{aligned} \quad (3)$$

with $f_1(0,0,0) = 0, Df_1(0,0,0) = 0$
 $f_2(0,0,0) = 0, Df_2(0,0,0) = 0$
 $f_3(0,0,0) = 0, Df_3(0,0,0) = 0$

Where f_1, f_2, f_3 are some $C^r, (r \geq 2)$ in some neighbourhood of the origin, A^c, A^s and A^u on the blocks are in the canonical form whose diagonals contain the eigenvalues with $\operatorname{Re} \lambda = 0, \operatorname{Re} \lambda < 0$ and $\operatorname{Re} \lambda > 0$, respectively, $(x, y, z) \in R^c \times R^s \times R^u, c = \dim E^c$ since the system (3) has c eigenvalues with zero real part, $s = \dim E^s$ and $u = \dim E^u, f_1, f_2$ and f_3 vanish along with their first partial derivatives at the origin.

If we assume that the unstable manifold is empty, then (3) becomes

$$\begin{aligned} \dot{x} &= A^c x + f_1(x, y) \\ \dot{y} &= A^s y + f_2(x, y) \end{aligned} \quad (4)$$

where $f_1(0,0) = 0, Df_1(0,0) = 0, f_2(0,0) = 0, Df_2(0,0) = 0,$

A^c is a $c \times c$ matrix having eigenvalues with zero real parts, A^s is an $s \times s$ matrix having eigenvalues with negative real parts, and f_1 and f_2 are C^r functions ($r \geq 2$.)

The following theorems established by Carr [24] in fact forms the basis of our discussion on the role of Centre manifold theorem in Hopf bifurcation and so we have stated them below:

A. Theorem 3.1:

There exists a C^r center manifold,

$$W^c = \{(x, y) | y = h(x), |x| < \delta, h(0) = 0, Dh(0) = 0\},$$

for δ sufficiently small, for (4) such that the dynamics of (4) restricted to the center manifold is given by the c -dimensional vector field

$$\dot{u} = A^c u + f_1(u, h(u)) \quad (5)$$

B. Theorem 3.2:

- (1) Suppose the zero solution of (5) is stable (asymptotically stable) (unstable) then the zero solution of (6) is also stable (asymptotically stable) (unstable).
- (2) Suppose the zero solution of (5) is stable. Then if $(x(t), y(t))$ is a solution of (4) with $(x(0), y(0))$ sufficiently small, there is a solution $u(t)$ of (5) such that as $t \rightarrow \infty$

$$x(t) = u(t) + O(e^{-\gamma t})$$

$$y(t) = h(u(t)) + O(e^{-\gamma t})$$

where $\gamma > 0$ is a constant.

From the theorem 3.2 it is clear that the dynamics of (5) near $u = 0$ determine the dynamics of (4) near $(x, y) = (0,0)$. To calculate $h(x)$ substitute $y = h(x)$ in the second component of (5) and using the chain rule, we obtain

$$N(h(x)) = Dh(x)[A^c x + f_1(x, h(x))] - Ch(x) - g(x, h(x)) = 0 \quad (6)$$

with boundary conditions $h(0) = Dh(0) = 0$. This differential equation for h cannot be solved exactly in most cases, but its solution can be approximated arbitrarily closely as a Taylor series at $x = 0$.

C. Theorem 3.3:

If a function $\phi(x)$, with $\phi(0) = D\phi(0) = 0$, can be found such that $N(\phi(x)) = O(|x|^p)$ for some $p > 1$ as $|x| \rightarrow 0$ then it follows that

$$h(x) = \phi(x) + O(|x|^p) \text{ as } |x| \rightarrow 0$$

This theorem allows us to compute the center manifold to any desired degree of accuracy by solving (6) to the same degree of accuracy.

In the discussion above we have assumed the unstable manifold is empty at the bifurcation point. If we include the unstable manifold then we must deal with the system (3). In this case $(x, y, z) = (0, 0, 0)$ is unstable due to the existence of a u -dimensional unstable manifold. However, much of the center manifold theory still applies, in particular Theorem 3.1 concerning existence, with the center manifold being locally represented by

$$W^c(0) = \{(x, y, z) \in R^c \times R^s \times R^u | y = h_1(x), z = h_2(x), h_i(0) = 0, Dh_i(0) = 0, i = 1, 2\}$$

for x sufficiently small. The vector field restricted to the center manifold is given by

$$\dot{u} = Au + f(x, h_1(u), h_2(u))$$

IV. HOPF BIFURCATION AND NORMAL FORM

One of the basic tools in the study of dynamical behavior of a system governed by nonlinear differential equations near a bifurcation point is the theory of normal forms. Normal form theory has been widely used in the study of nonlinear vector fields in order to simplify the analysis of the original system [3, 19, 26, 34, 36, 41, 51, 52, 55].

Several efficient methodologies for computing normal forms have been developed in the past decade [1, 27, 35, 39, 42, 43, 44, 45, 46, 57].

The method of normal forms can be traced back to the Ph.D thesis of Poincare. The books by Van der Meer [29] and Bryuno [2] give valuable historical background.

The basic idea of normal form theory is to employ successive, near identity nonlinear transformations to eliminate the so called non-resonant nonlinear terms, and retaining the terms which cannot be eliminated (called resonant terms) to form the normal form and which is sufficient for the study of qualitative behavior of the original system.

The Local Center Manifold Theorem in the previous section showed us that, in a neighborhood of a non-hyperbolic critical point, determining the qualitative behavior of (3) could be reduced to the problem of determining the qualitative behavior of the nonlinear system

$$\dot{x} = A^c x + F(x) \quad (7)$$

on the center manifold. Since the dimension of the center manifold is typically less than n , this simplifies the problem of determining the qualitative behavior of the system (1) near a non-hyperbolic critical point. However, analyzing this system still may be a difficult task. The normal form theory allows us to simplify the nonlinear part, $F(x)$ of (7) in order to make this task as easy as possible. This is accomplished by making a nonlinear, analytic transformation (called near identity transformation) of coordinates of the form

$$x = y + h(y), \text{ where } h(y) = O(|y|^2) \text{ as } |y| \rightarrow 0. \quad (8)$$

Suppose that $Df_\mu(x_0)$ has two purely imaginary eigenvalues with the remaining $(n - 2)$ eigenvalues having non-zero real parts. We know that since the fixed point is not hyperbolic, the orbit structure of the linearised vector field near $(x, \mu) = (x_0, \mu_0)$ may reveal little (and possibly even incorrect) information concerning the nature of the orbit structure of the nonlinear vector field (1) near $(x, \mu) = (x_0, \mu_0)$. But by the center manifold theorem, we know that the orbit structure near $(x, \mu) = (x_0, \mu_0)$ is determined by the vector field (1) restricted to the center manifold.

On the center manifold, the vector field (3) has the following form

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} Re \lambda(\mu) & -Im \lambda(\mu) \\ Im \lambda(\mu) & Re \lambda(\mu) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} f^1(x, y, z) \\ f^2(x, y, z) \end{pmatrix} \quad (x, y, \mu) \in \mathbb{R}^1 \times \mathbb{R}^1 \times \mathbb{R}^1 \quad (9)$$

where f^1 and f^2 are nonlinear in x and y and $\lambda(\mu), \overline{\lambda(\mu)}$ are the eigenvalues of the vector field linearized about the fixed point at the origin.

Now, if we denote the eigenvalue

$$\lambda(\mu) = \alpha(\mu) + i\omega(\mu), \quad (10)$$

then, for our assumption of non hyperbolic nature we have

$$\alpha(0) = 0, \quad \omega(0) \neq 0 \quad (11)$$

Now, we transform the equation (9) to the normal form. The normal form is found to be

$$\begin{aligned} \dot{x} &= \alpha(\mu)x - \omega(\mu)y + (l(\mu)x - m(\mu)y)(x^2 + y^2) + O(|x|^5, |y|^5) \\ \dot{y} &= \omega(\mu)x + \alpha(\mu)y + (m(\mu)x + l(\mu)y)(x^2 + y^2) + O(|x|^5, |y|^5) \end{aligned} \quad (12)$$

In polar coordinates the equation (12) can be written as

$$\begin{aligned} \dot{r} &= \alpha(\mu)r + l(\mu)r^3 + O(r^5), \\ \dot{\theta} &= \omega(\mu) + m(\mu)r^2 + O(r^4) \end{aligned} \quad (13)$$

Since we are interested in the dynamics near $\mu = 0$, therefore expanding in Taylor's series the coefficients in (13) about $\mu = 0$, the equation (13) becomes

$$\begin{aligned} \dot{r} &= \alpha'(0)\mu r + l(0)r^3 + O(\mu^2 r, \mu r^3, r^5), \\ \dot{\theta} &= \omega(0) + \omega'(0)\mu + m(0)r^2 + O(\mu^2, \mu r^2, r^4) \end{aligned} \quad (14)$$

where “ ’ ” denotes differentiation with respect to μ and we have used the fact that $\alpha(0) = 0$.

Our goal is to understand the dynamics of (14) for small r and μ . This is accomplished in two steps. In the first step, we neglect the higher order terms of (14) to get a “truncated” normal form and in the second step we study the dynamics exhibited by the truncated normal form as it is well known that the dynamics exhibited by the truncated normal form are qualitatively unchanged when one considers the influence of the previously neglected higher order terms [53].

Now, neglecting the higher order terms in (14) gives

$$\begin{aligned} \dot{r} &= d\mu r + lr^3, \\ \dot{\theta} &= \omega + c\mu + mr^2 \end{aligned} \quad (15)$$

where for ease of notation, we have

$$\alpha'(0) \equiv d, \quad l(0) \equiv l, \quad \omega(0) \equiv \omega, \quad \omega'(0) \equiv c, \quad m(0) \equiv m$$

For $-\infty < \frac{\mu d}{l} < 0$ and μ sufficiently small, the solution of (17) is given by

$$(r(t), \theta(t)) = \left(\sqrt{-\frac{\mu d}{l}}, \left[\omega + \left(c - \frac{md}{l} \right) \mu \right] t + \theta_0 \right)$$

which is a periodic orbit for (15) and the periodic orbit is asymptotically stable for $l < 0$ and unstable for $l > 0$ [26, 53].

In practice, l is straight forward to calculate. It can be calculated simply by keeping track of the coefficients carefully in the normal form transformation in terms of the original vector field. The expression of l for a two dimensional system of the form

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} f(x, y) \\ g(x, y) \end{pmatrix}$$

with $f(0) = 0 = g(0) = 0$, is found to be [35, 25, 31]

$$l = \frac{1}{16} (f_{xxx} + f_{xyy} + g_{xxy} + g_{yyy}) + \frac{1}{16\omega} (f_{xy}(f_{xx} + f_{yy}) - g_{xy}(g_{xx} + g_{yy}) - f_{xx}g_{xx} + f_{yy}g_{yy})$$

In fact, all the above discussions get converged in the famous Hopf Bifurcation Theorem which states as follows:

Let the eigenvalues of the linearized system of $\dot{x} = f_\mu(x)$, $x \in R^n$, $\mu \in R$ about the equilibrium point be given by $\lambda(\mu), \bar{\lambda}(\mu) = \alpha(\mu) \pm i\beta(\mu)$. Suppose further that for a certain value of μ say $\mu = \mu_0$, the following conditions are satisfied:

- (1) $\alpha(0) = 0, \beta(0) = \omega \neq 0$
- (2) $\left. \frac{d\alpha(\mu)}{d\mu} \right|_{\mu=\mu_0} = d \neq 0$
(transversality condition : the eigenvalues cross the imaginary axis with nonzero speed)
- (3) $l \neq 0$, where

$$l = \frac{1}{16} (f_{xxx} + f_{xyy} + g_{xxy} + g_{yyy}) + \frac{1}{16\omega} (f_{xy}(f_{xx} + f_{yy}) - g_{xy}(g_{xx} + g_{yy}) - f_{xx}g_{xx} + f_{yy}g_{yy})$$

where f_{xy} denotes $\left. \frac{\partial^2 f_\mu}{\partial x \partial y} \right|_{\mu=\mu_0} (x_0, y_0)$, etc.

(genericity condition)

then there is a unique three-dimensional center manifold passing through (x_0, μ_0) in $R^n \times R$ and a smooth system of coordinates (preserving the planes $\mu = \text{const.}$) for which the Taylor expansion of degree 3 on the center manifold is given by

(12). If $l \neq 0$, there is a surface of periodic solutions in the center manifold which has quadratic tangency with the eigenspace of $\lambda(\mu_0), \bar{\lambda}(\mu_0)$ agreeing to second order with the paraboloid $\mu = -(l/d)(x^2 + y^2)$. If $l < 0$, then these periodic solutions are stable limit cycles, while if $l > 0$, the periodic solutions are repelling (unstable limit cycles).

V. DYNAMICAL ANALYSIS

A. Equations:

The dynamics of the Brusselator reaction can be described by a system of two ODE's. In dimensionless form, they are:

$$\begin{aligned} \dot{x} &= a - (1 + b)x + yx^2 \\ \dot{y} &= bx - yx^2 \end{aligned} \quad (16)$$

B. Analysis of equilibrium point :

The equilibria of (16) are given by solving the system

$$a - (1 + b)x + yx^2 = 0 \quad (17)$$

$$bx - yx^2 = 0 \quad (18)$$

From (17) and (18), we get

$$\begin{aligned} x &= a \quad \text{and} \quad y = \frac{b}{a} \\ \therefore \left(a, \frac{b}{a}\right) &\text{ is the only equilibrium point.} \end{aligned}$$

Now, the Jacobian of the system is

$$J = \begin{pmatrix} -b - 1 + 2xy & a^2 \\ b - 2xy & a^2 \end{pmatrix} \quad (19)$$

$$\text{At} \left(a, \frac{b}{a}\right), J = \begin{pmatrix} b - 1 & a^2 \\ -b & -a^2 \end{pmatrix} \quad (20)$$

The eigenvalues of J are given by

$$\lambda = \frac{(b - a^2 - 1) \pm i\sqrt{4a^2 - (b - a^2 - 1)^2}}{2} \quad (21)$$

The equilibrium is an attractor if $Re(\lambda) < 0$ i.e., $b < a^2 + 1$ whereas the equilibrium is a repeller if $Re(\lambda) > 0$ i.e., $b > a^2 + 1$. The equilibrium point $\left(a, \frac{b}{a}\right)$ is not a saddle point as the determinant $D = a^2 > 0$. Below, we have shown the regions of the parameters a, b for which we will have stable and unstable fixed points

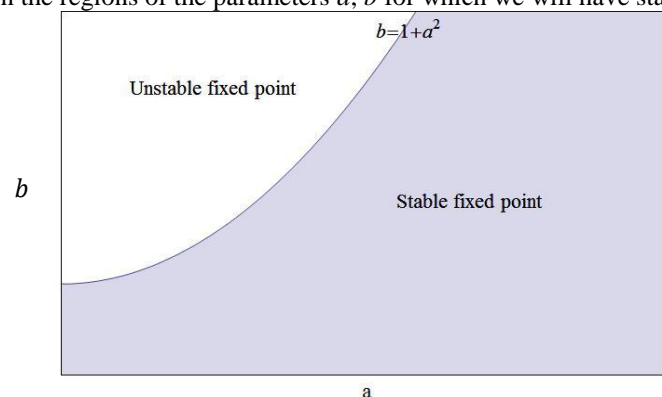


Fig. 1: Stability Analysis in Parameter Space

To determine the solution of the characteristic polynomial $\lambda^2 - T\lambda + D = 0$, we compute the trace (T) and the determinant (D) of J ,

$$T = b - 1 - a^2$$

$$D = a^2$$

$$T^2 - 4D = (b - (a + 1)^2)(b - (a - 1)^2)$$

For $T > 0$ the fixed point is unstable, while for $T < 0$ it is stable. Changing the stability at $T = 0$, we find the critical parameter values by the equation $b = 1 + a^2$. The determinant D is always positive.

$T^2 - 4D$ is negative if $(a - 1)^2 < b < (a + 1)^2$ and positive otherwise.

The possible behaviours of the system can be classified as a function of parameters a and b as shown below:

Type of steady state	Stable node		Stable focus		Unstable focus		Unstable node
b		$(a - 1)^2$		$a^2 + 1$		$(a + 1)^2$	
T	< 0	< 0	< 0	0	> 0	> 0	> 0
D	> 0	> 0	> 0	> 0	> 0	> 0	> 0
$T^2 - 4D$	> 0	0	< 0	< 0	< 0	0	> 0

When parameter b increases, the steady state turns from stable node to a stable focus, then it loses its stability (the system evolves towards a limit cycle) and the steady state turns from an unstable focus to an unstable node. The stable and unstable regions as a function of a and b is shown in fig.1

The dotted curve corresponding to $b = a^2 + 1$ portrays the stability region. The solid curves corresponding to $b = (a + 1)^2$ and $b = (a - 1)^2$ separate the node from the focus in the stable and unstable region respectively.

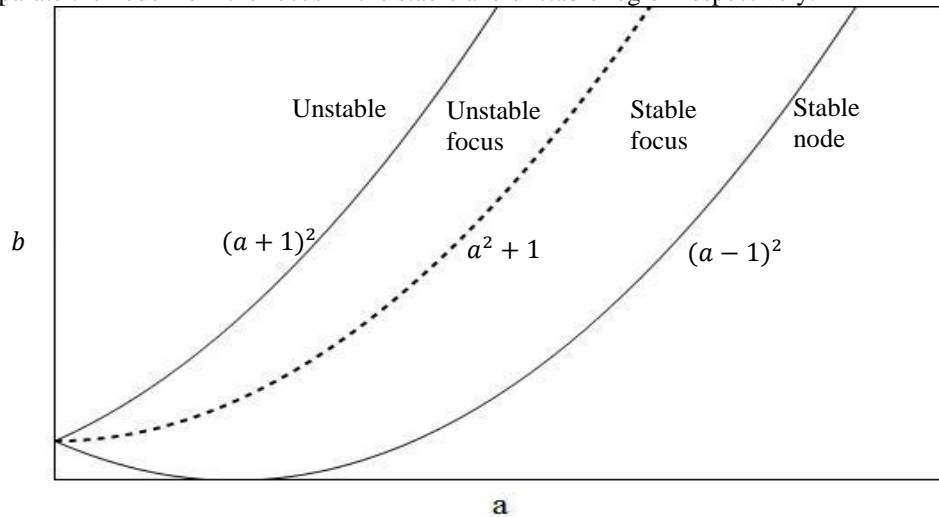


Fig. 2: Stability Diagram For The Brusselator Model

C. Nullclines:

The method of nullclines [32, 50] is a technique for determining global behaviour of solutions. This method provides an effective means of finding trapping regions for some differential equations.

The \dot{x} -nullcline (shown as red curve) is given by setting $\dot{x} = 0$, which in case of our system becomes

$$a - (1 + b)x + yx^2 = 0$$

$$\Rightarrow y = \frac{(1+b)x-a}{x^2} \tag{22}$$

The \dot{y} -nullcline (shown as blue curve) is given by setting $\dot{y} = 0$, which in case of our system becomes

$$bx - yx^2 = 0$$

$$\Rightarrow x(b - xy) = 0$$

$$\Rightarrow x = 0 \text{ or}$$

$$y = \frac{b}{x} \tag{23}$$

Thus, the \dot{y} -nullcline consists of the y-axis and a hyperbola.

For ($a = 1, b = 2.5$) the nullclines are as shown in Figure below:

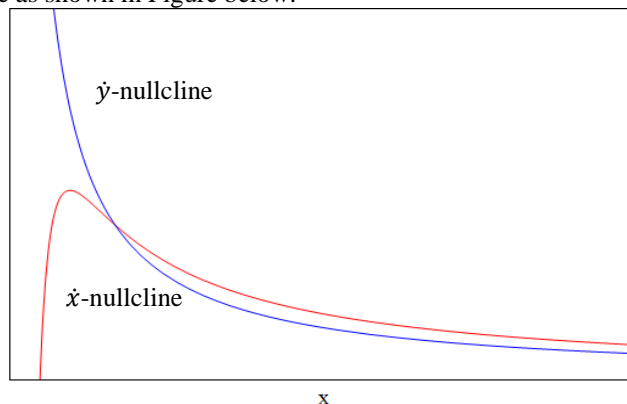


Fig. 3: The Nullclines of The Brusselator Model For The Parameter Values $a = 1, b = 2.5$.

In the figure below, the nullclines are shown along with some representative vectors.

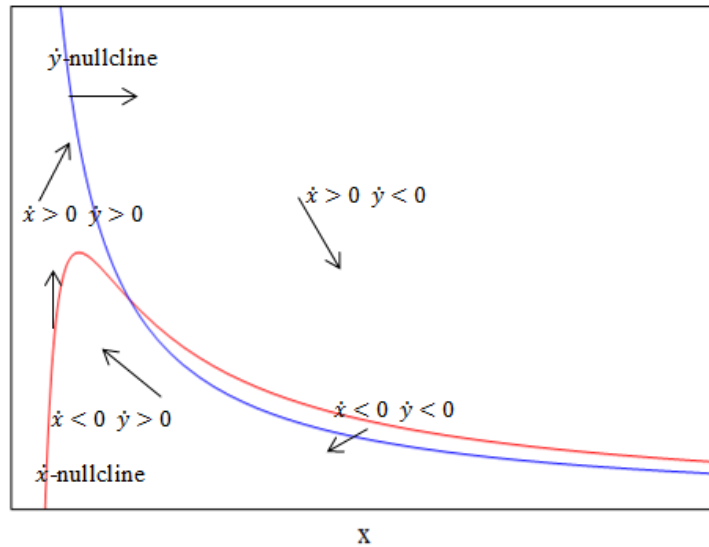


Fig. 4: The Nullclines With Some Representative Vectors

D. Trapping region:

To establish the existence of periodic orbit, we need to construct a trapping region. Below we have discussed the technique of finding the trapping region [32, 50].

We have already shown that when $b > a^2 + 1$ then the equilibrium point $(a, \frac{b}{a})$ is a repellor. In this case, surround the equilibrium point by a small circular region of radius $\delta > 0$. Further, assume that for $\delta > 0$ all vectors on the boundary of δ -ball centred at $(a, \frac{b}{a})$ point to exterior.

Now, consider the nullclines and the region bounded by them in the first quadrant. We can use the x-axis as the lower boundary. The left boundary will need to intercept the x-axis at the point where the \dot{x} -nullcline intersects it. Now, the x intercept of the \dot{x} -nullcline is given by

$$a - (1 + b)x + 0 = 0$$

$$\Rightarrow x = \frac{a}{1+b} \tag{24}$$

Thus, the left boundary of the trapping region is found to be

$$x = \frac{a}{1+b}.$$

It is to be noted that across this left boundary we have $\dot{x} > 0$.

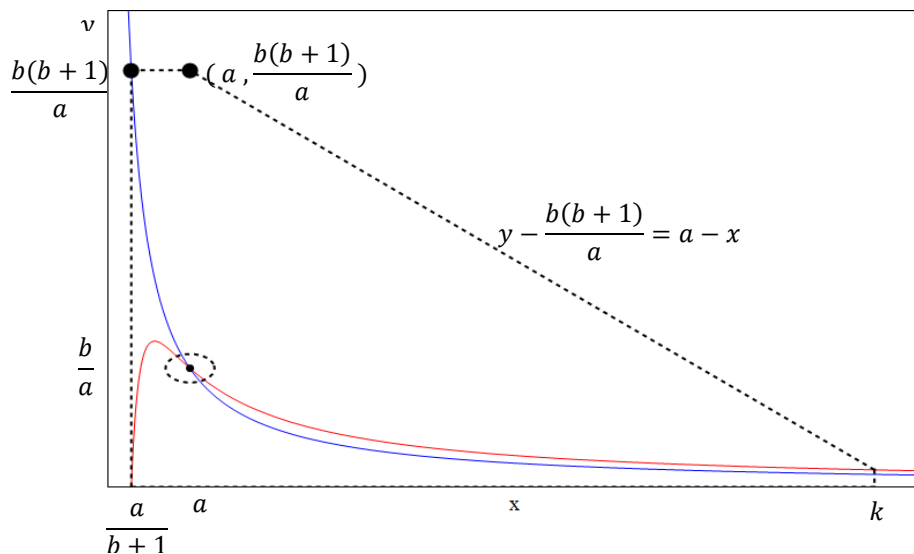


Fig. 5: Trapping Region

Next, to fix the upper boundary, we look for the point of intersection of the left boundary with the \dot{y} -nullcline which is found by putting $x = \frac{a}{1+b}$ into the equation of \dot{y} -nullcline given by equation (23). Thus, the upper boundary is found to be

$$y = \frac{b(b+1)}{a} \quad (25)$$

It is to be noted that across this upper boundary we have $\dot{y} < 0$.

It is further to be noted that as $b > a^2 + 1$ so $y > a$ for the upper boundary.

Construction of the next boundary is little bit tricky here. We have to construct this boundary segment of the trapping region in such a way that the flow across this boundary will be directed towards the interior of the trapping region. Therefore, for a normal vector $n = (n_1, n_2)$ to the segment we require that

$$(n_1, n_2) \cdot (\dot{x}, \dot{y}) = n_1\dot{x} + n_2\dot{y} < 0 \quad (26)$$

For convenience, we choose $n_1 = n_2 = 1$ and see that $\dot{x} + \dot{y} = a - x$. Therefore, to get $\dot{x} + \dot{y} < 0$ we must have $x > a$.

Thus, as long as $x > a$ a segment with normal vector $(1, 1)$ will have the desired property.

Thus, our segment will be defined by the line

$$y - \frac{b(b+1)}{a} = a - x$$

Next, we have to find our right boundary of the trapping region for which we must have $\dot{x} < 0$ for any point below the \dot{x} nullcline. This can be easily constructed by using the x coordinate of the intersection of the above line segment and \dot{x} nullcline. It is to be noted that that this right boundary of the trapping region will be given by the vertical line $x = k$ where k is the x coordinate of the the point of intersection.

Having constructed a trapping in which there is no equilibrium, we can apply the Poincare Bendixson theorem to establish the existence of periodic solution somewhere within the region

E. Poincare-Bendixson Theorem:

Let D be a closed bounded region of the $x - y$ plane and

$$\dot{x} = f(x, y)$$

$$\dot{y} = g(x, y)$$

be a dynamical system in which f and g are continuously differentiable. If a trajectory of the dynamical system is such that it remains in D for all $t \geq 0$ then the trajectory must

Be a closed orbit or

approach a closed orbit or

approach an equilibrium point as $t \rightarrow \infty$

The implication of this theorem is that if we can find a trapping region for a dynamical system which does not contain an equilibrium point then there must be at least one limit cycle within the region. Also, since the periodic solution appears when equilibrium destabilizes, this Hopf bifurcation is supercritical.

In case of our considered system, the above mentioned required conditions are found to be satisfied and hence we can conclude that we must have a supercritical Hopf bifurcation inside the trapping region which we found out in section d.

F. Bifurcation analysis:

The equilibrium point $(a, \frac{b}{a})$ undergoes a change in stability as 'b' varies when 'a' is kept fixed. The eigenvalues of (20) are purely imaginary when $b_c = a^2 + 1$.

Again From (21), we have

$$\begin{aligned} Re(\lambda) &= \frac{b-a^2-1}{2} \\ \therefore \frac{\partial}{\partial b} Re(\lambda) &= \frac{1}{2} \neq 0 \end{aligned} \quad (27)$$

Thus, the Hopf bifurcation conditions are satisfied in case of our considered system and the Hopf bifurcation occurs at $b_c = a^2 + 1$.

VI. NORMAL FORM

Below, we have justified our earlier claim about the occurrence of a supercritical Hopf bifurcation in our considered system

$$\dot{x} = a - (1 + b)x + yx^2$$

$$\dot{y} = bx - yx^2$$

with the help of Normal form.

The Jacobian, J of the system at $\left(a, \frac{b}{a}\right)$

$$J = \begin{pmatrix} b-1 & a^2 \\ -b & -a^2 \end{pmatrix}$$

The eigenvalues are given by

$$\lambda = \frac{(b - a^2 - 1) \pm i\sqrt{4a^2 - (b - a^2 - 1)^2}}{2}$$

From above it is seen that $\lambda = \pm ia$ at $b = 1 + a^2$ which verifies that a Hopf bifurcation occurs $b = 1 + a^2$.

To study the stability nature of Hopf bifurcation, we change the coordinate twice: first to bring the point $\left(a, \frac{b}{a}\right)$ to the origin and then to put the vector field into standard form.

Letting $\bar{x} = x - a$ and $\bar{y} = y - \frac{b}{a}$ we have

$$\begin{pmatrix} \dot{\bar{x}} \\ \dot{\bar{y}} \end{pmatrix} = \begin{pmatrix} b-1 & a^2 \\ -b & -a^2 \end{pmatrix} \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} + \begin{pmatrix} \frac{b}{a}\bar{x}^2 + \bar{y}\bar{x}^2 + 2a\bar{x}\bar{y} \\ -\frac{b}{a}\bar{x}^2 - \bar{x}^2\bar{y} - 2a\bar{x}\bar{y} \end{pmatrix}$$

The coefficient matrix of the linear part is

$$A = \begin{pmatrix} b-1 & a^2 \\ -b & -a^2 \end{pmatrix}$$

The eigenvalues are $\lambda = \pm ia$

Therefore, the eigenvectors corresponding to the above eigenvalues are $\begin{pmatrix} 1 \\ \pm ia \end{pmatrix}$

Then, using the linear transformation $\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} = T \begin{pmatrix} u \\ v \end{pmatrix}$

where $T = \begin{pmatrix} 0 & a^2 \\ a & -b+1 \end{pmatrix}$ is the matrix of real and imaginary parts of the eigenvectors of the eigenvalues $\lambda = \pm ia$, we obtain the system in standard form as :

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 0 & -a \\ a & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} (-3 + 3b + a^2b - 3a)v^2 \\ (2 - b)av^2 + a^3uv^2 + (1 - b)a^2v^3 + 2a^2uv \end{pmatrix}$$

Now, let us consider the nonlinear parts of the above form as

$$f(u, v) = (-3 + 3b + a^2b - 3a)v^2$$

$$g(u, v) = (2 - b)av^2 + a^3uv^2 + (1 - b)a^2v^3 + 2a^2uv$$

Now, following [15, 16, 24], we calculate

$$f_u = 0, f_{uu} = 0, f_{uuu} = 0, f_{uv} = 1, f_{uvv} = 0, f_v = 0, f_{vv} = 2(-3 + 3b + a^2b - 3a^2)$$

$$g_u = 0, g_{uu} = 0, g_{uuv} = 0, g_v = 0, g_{vv} = 0, g_{vvv} = 6(1 - b)a^2, g_{uv} = 2a^2$$

and obtain

$$l = \frac{1}{16} [f_{uuu} + f_{uvv} + g_{uuu} + g_{vvv}] + \frac{1}{16\omega} [f_{uv}(f_{uu} + f_{vv}) - g_{uv}(g_{uu} + g_{vv}) - f_{uu}g_{uu} + f_{vv}g_{vv}]$$

$$= -\frac{a^2(2+a^2)}{8} < 0$$

which shows that the Hopf bifurcation is supercritical.

VII. CONCLUSION

In this paper we have investigated the stability nature of Hopf bifurcation in a two dimensional nonlinear differential equation, popularly known as the Brusselator model. We have done dynamical analysis of the model, formed the trapping region with the help of nullclines, justified the existence of Hopf bifurcation with the help of Poincare Benedixson theorem and Hopf bifurcation theorem and used the technique of Normal forms to show that supercritical Hopf bifurcation occurs in the system we have considered.

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