Flexural Response of Functionally Graded Plates Based on a Higher Order Shear Deformation Theory Under Point Load

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Abstract

A flexural response of functionally graded material (FGM) plates is carried out using multi quadric radial based functional based meshless method. The governing equations are based on the higher order shear deformation theory are obtained using energy principle. A RBF based meshless code in MATLAB 2013 is developed to find out the results. In present paper, the effects of significant parameters, such as gradation index, span to thickness ratio is obtained at transverse point load. The present paper mainly focuses on the analysis of FGM plates under point load.
Keywords: Point load, FGM, Plates, Meshless method, RBF, Flexural

I. INTRODUCTION

Recently functionally graded material (FGM) emerged rapidly in the field of composite structures. FGM is the non homogenous composite structure material know for their high strength-to-weight and savings of both cost and weight and achieved by gradually varying the volume fraction of the constituent materials in the thickness direction only. Thus material reduces thermal stresses, residual stresses and stress concentration factors in FGMs plates. The term FGMs was originated in the mid-1980s by a group of scientists in Japan. In last few years, FGM plates attracted researchers for analysis purpose and for the analyzing of plates many theories developed. The deflections of a simply supported FG polygonal plate based on the first-order shear deformation theory and a third-order shear deformation was investigated by Cheng [1]. Vel et al. [2] have carried an exact 3D solution of FG simply supported plates of finite dimensions for the thermoelastic deformation. Pan [3] introduced an exact solution for functionally anisotropic elastic composite laminates. Meshless method based on third-order shear deformation theory for static analysis of a simply supported functionally graded plate was carried by Ferreira et al [4] . Qian et al. [5] studied local Petrov–Galerkin meshless method for the static and dynamic deformation of thick FG elastic plates by applying higher order shear and normal deformable plate theories. Xiang et al. [6] carried Gaussian radial basis function and first order shear deformation theory to investigate natural frequencies of generally laminated composite plates. Mesh free method was used by Liu [7] et al. to investigated the studied free vibration analysis of thin plates of complicated shapes

II. MATHEMATICAL FORMULATION

A rectangular shape plate of edge length a, b along x, y axes respectively and thickness h is the thickness along z axis whose mid plane is coinciding with x-y plane of the coordinate system is considered. The diagram of rectangular shaped functionally graded material (FGM) plate in rectangular coordinate system is shown in Figure 1.

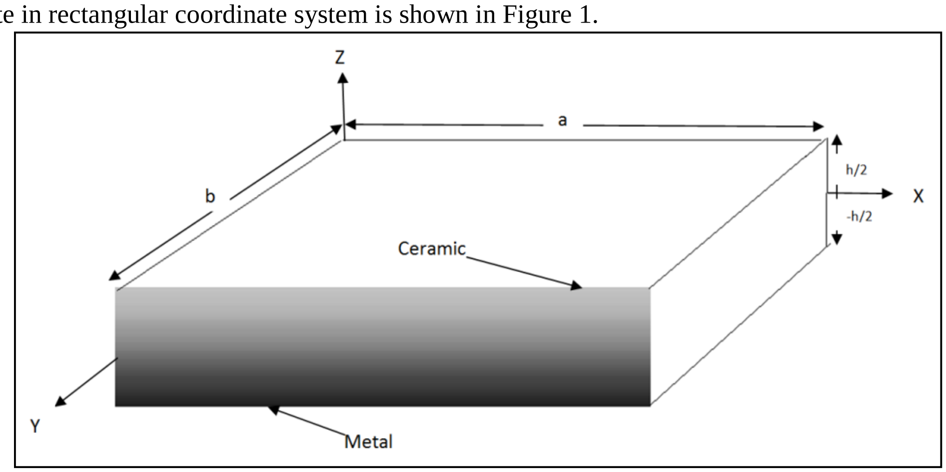

Fig. 1: Geometry of rectangular FGM plate in rectangular coordinate system

The homogenization technique considered in this work is the law of mixtures, which provides the following elastic properties at each material layer. The top surface of the plate is ceramic rich and the bottom surface is metal rich.

$$V_c(z) = \left(\frac{2z+h}{2h}\right)^n \quad (1)$$

Where 'n' is exponent governing the material properties along the thickness direction known as volume fraction exponent or grading index,
The volume fraction of the metal phase is obtained by

$$V_m(z) = 1 - V_c(z) \quad (2)$$

The material property gradation through the thickness of the plate is assumed to have the following form

$$E(z) = \left[E_c - E_m\right]\left(\frac{2z+h}{2h}\right)^n + E_m \quad (3)$$

Here *E* denote the modulus of elasticity of FGM structure, while these parameters come with subscript m or c represent the material properties for pure metal and pure ceramic plate respectively., h is the thickness of the plate, E_m and E_c are the corresponding Young's modulus of elasticity of metal and ceramic and z is the thickness coordinate.
The displacement field at any point in the plate made up of uniform thickness is expressed as:

$$\begin{Bmatrix} U_x \\ U_y \\ U_z \end{Bmatrix} = \begin{Bmatrix} u_x(x,y) - z\dfrac{\partial u_z(x,y)}{\partial x} + \left[\dfrac{5z}{4} - \dfrac{5}{3h^2}z^3\right]\psi_x(x,y) \\ u_y(x,y) - z\dfrac{\partial u_z(x,y)}{\partial y} + \left[\dfrac{5z}{4} - \dfrac{5}{3h^2}z^3\right]\psi_y(x,y) \\ u_z(x,y) \end{Bmatrix} \quad (4)$$

The constitutive stress-strain relations for any FGM plate are expressed as:

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{Bmatrix} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & 0 & 0 & 0 \\ \bar{Q}_{12} & \bar{Q}_{22} & 0 & 0 & 0 \\ 0 & 0 & \bar{Q}_{66} & 0 & 0 \\ 0 & 0 & 0 & \bar{Q}_{44} & 0 \\ 0 & 0 & 0 & 0 & \bar{Q}_{55} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix}$$

Where, the parameters Q_{ij} are the stiffness coefficients and are expressed in terms of elastics constants as:

$$\bar{Q}_{11} = \bar{Q}_{22} = \frac{E}{\left(1-\nu^2\right)}, \quad \bar{Q}_{12} = \frac{\nu E}{\left(1-\nu^2\right)}, \bar{Q}_{44} = \bar{Q}_{55} = \bar{Q}_{66} = G$$

The governing differential equations of plate are obtained using energy equation, in mathematical form it is expressed as:

$$\int_{t_1}^{t_2} \delta\left(U+V\right)dt = 0 \quad (5)$$

Where, U = Strain energy
V = workdone due to transverse load
The strain energy of the plate due to internal stress resultants is expressed as:

$$U = \frac{1}{2}\int_{Volume}\left(\sigma_{xx}\varepsilon_{xx} + \sigma_{yy}\varepsilon_{yy} + \sigma_{xy}\gamma_{xy} + \sigma_{yz}\gamma_{yz} + \sigma_{xz}\gamma_{xz}\right)dxdydz \quad (6)$$

$$V = \int_{Area} u_z q_z \, dx dy \qquad (7)$$

The governing differential equations of plate are obtained using Hamilton's principle and expressed as :

$$\frac{\partial N_{xx}}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0$$

$$\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_{yy}}{\partial y} = 0$$

$$\frac{\partial^2 M_{xx}}{\partial x^2} + \frac{\partial^2 M_{yy}}{\partial y^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + q_z = 0$$

$$\frac{\partial M_{xx}^f}{\partial x} + \frac{\partial M_{xy}^f}{\partial y} - Q_x^f = 0 \qquad (8)$$

$$\frac{\partial M_{xy}^f}{\partial x} + \frac{\partial M_{yy}^f}{\partial y} - Q_y^f = 0$$

The force and moment resultants in the plate and plate stiffness coefficients are expressed as:

$$N_{ij},\ M_{ij},\ M_{ij}^f = \int_{-h/2}^{+h/2} \left(\sigma_{ij},\ z\sigma_{ij},\ \left[\frac{5z}{4} - \frac{5}{3h^2} z^3\right]\sigma_{ij}\right) dz \qquad (9)$$

$$Q_x^f,\ Q_y^f = \int_{-h/2}^{+h/2} \left(\sigma_{xz},\ \sigma_{yz}\right)\left(\frac{\partial\left[\frac{5z}{4} - \frac{5}{3h^2} z^3\right]}{\partial z}\right) dz \qquad (10)$$

$$A_{ij}, B_{ij}, D_{ij}, E_{ij}, F_{ij}, H_{ij} = \int_{-h/2}^{h/2} \left\{ Q(z) \times \left(1, z, z^2, \left[\frac{z}{4} - \frac{5}{3h^2} z^3\right], z \times \left[\frac{5z}{4} - \frac{5}{3h^2} z^3\right], \left[\frac{5z}{4} - \frac{5}{3h^2} z^3\right]^2\right)\right\} dz \qquad (11)$$

i, j = 1, 2, 6

$$A_{ij} = \int_{-h/2}^{h/2} \left\{ Q(z) \times \left(\frac{\partial\left[\frac{5z}{4} - \frac{5}{3h^2} z^3\right]}{\partial z}\right)^2 \right\} dz \qquad (12)$$

i, j = 4, 5

$$where, \quad Q(z) = \left(\left(\left[Q_{ij}^c - Q_{ij}^m \right] \right) \left(\frac{2z + h}{2h} \right)^n + Q_{ij}^m \right)$$

The boundary conditions for an arbitrary edge with simply supported conditions are as follows:

$$x = 0, a : u_y = 0; \psi_y = 0; u_z = 0; M_{xx} = 0; N_{xx} = 0$$

$$y = 0, b : u_x = 0; \psi_x = 0; u_z = 0; M_{yy} = 0; N_{yy} = 0$$

III. SOLUTION METHODOLOGY

The governing differential equations (8) are expressed in terms of displacement functions. Radial basis function based formulation works on the principle of interpolation of scattered data over entire domain. A 2D rectangular domain having NB boundary nodes and ND interior nodes is shown in Figure-2.

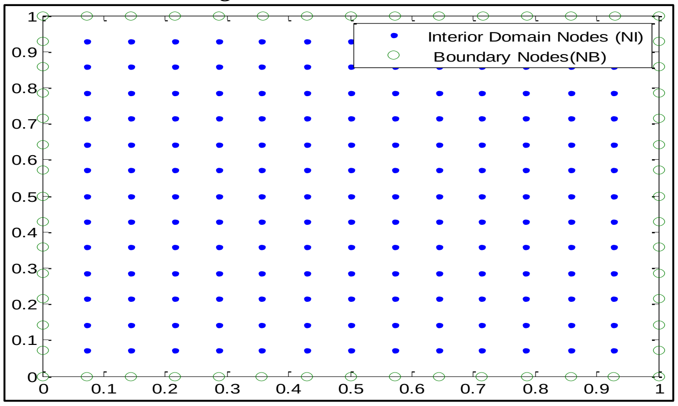

Fig. 2: An arbitrary two dimensional domains

The variable $u_x, u_y, u_z, \psi_x, \psi_y$ can be interpolated in form of radial distance between nodes. The solution of the linear governing differential equations (8) is assumed in terms of multiquadric radial basis function for nodes 1:N, as

$$u_x, u_y, u_z, \psi_x, \psi_y = \sum_{j=1}^{N} (\alpha_j^{u_x}, \alpha_j^{u_y}, \alpha_j^{u_z}, \alpha_j^{\psi_x}, \alpha_j^{\psi_y})\ g\left(\left\|X - X_j\right\|, m, c\right)$$

Where, N is total numbers of nodes which is equal to summation of boundary nodes NB and domain interior nodes ND. $g\left(\left\|X - X_j\right\|, m, c\right)$ is multiquadric radial basis function expressed as $g = \left(r^2 + c^2\right)^m$ $,(\alpha_j^{u_x}, \alpha_j^{u_y}, \alpha_j^{u_z}, \alpha_j^{\psi_x}, \alpha_j^{\psi_y})$ are unknown coefficients. $\left\|X - X_j\right\|$ is the radial distance between two nodes.

Where, $r = \left\|X - X_j\right\| = \sqrt{\left(x - x_j\right)^2 + \left(y - y_j\right)^2}$ and m and c are shape parameters. The value of 'm' and 'c' taken here is 0.5 and 1.3/(N)0.25.

IV. COMPUTATION AND DISCUSSION OF RESULTS

The study here has been focused on the flexural response of simply supported square functionally graded plates under point loads. A RBF based meshless code in MATLAB 2013 is developed. Several examples have been analyzed and the computed results are compared. Based on convergence study, a 15×15 node is used throughout the study. The material properties of FGMs have been taken as follows

Ceramic $E_c = 151 GPa,\ \ \nu_c = 0.3$

Aluminum (Al) $E_m = 70 GPa,\ \ \nu_m = 0.3$

In order to show the accuracy and efficiency of the present solution methodology, detailed convergence studies for simply supported FGM plate (a/h=20) is carried out. The convergences of the deflection are shown in Fig. 3. It can be seen that convergence achieved is within 1 % at 15×15 nodes.

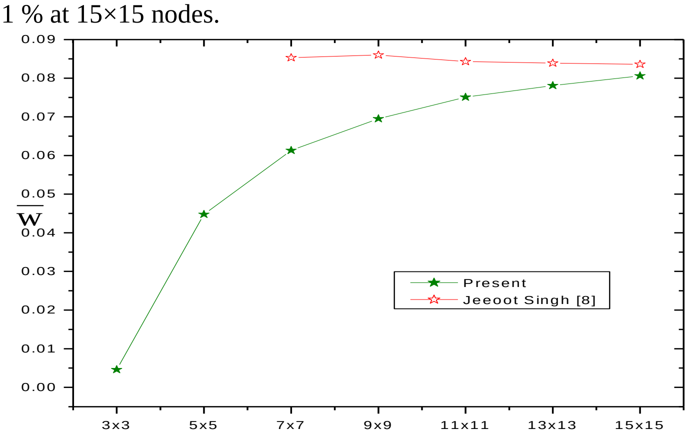

Fig. 3: Convergence study for deflection of a simply supported FGM plate (a/h = 20, ‘n’=2)

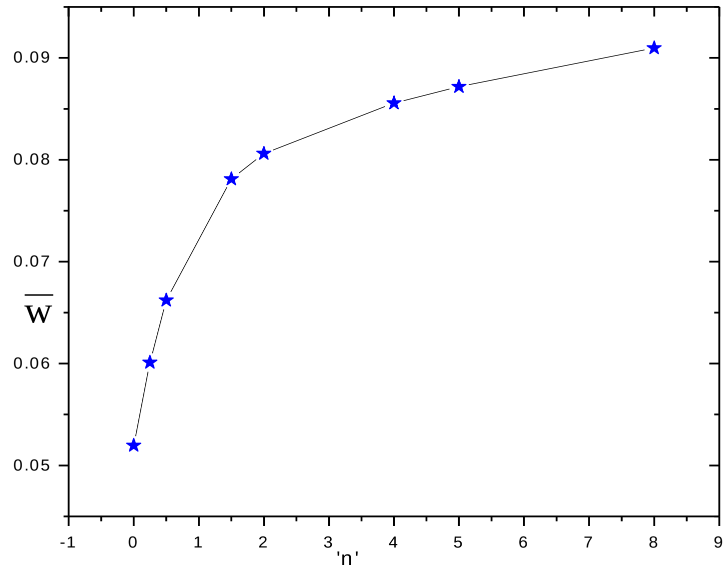

Fig. 4: Effect of grading index 'n' on deflection of a square FGM plate (a/h=20)

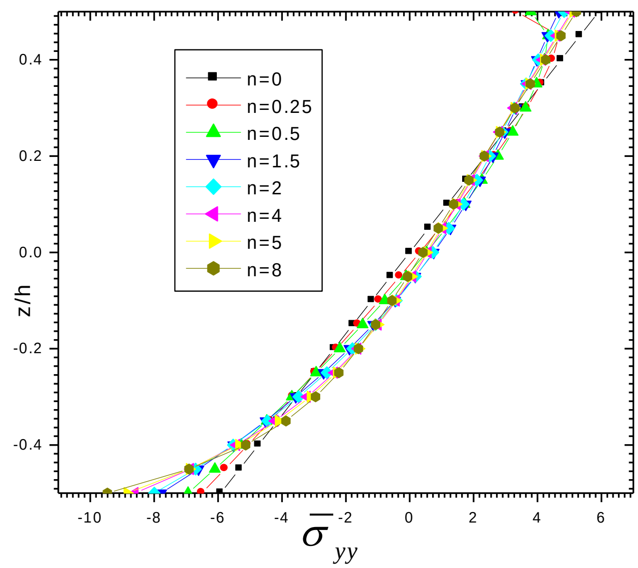

Fig. 5: Effect of grading index 'n' on normalized stress of simply supported square FGM plate along the thickness

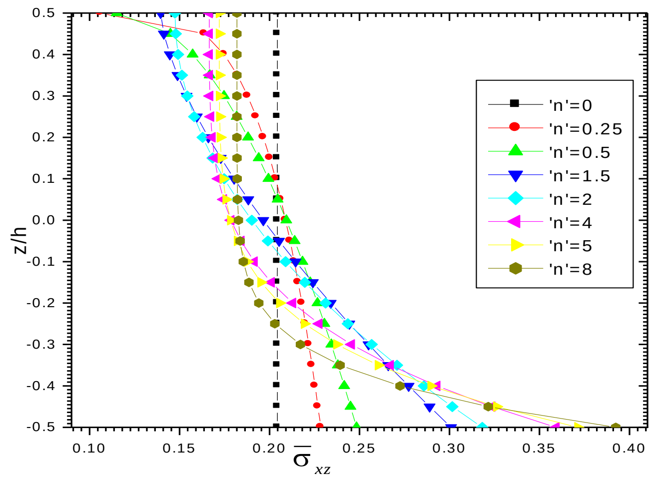

Fig. 6: Effect of grading index 'n' on normalized stress of simply supported square FGM plate along the thickness

Table – 1
Effect of gradation index 'n' on deflection, stresses and Moments of a simply supported FGM Plate(a/h=5)

	'n'								
	0	0.25	0.5	0.75	1	2	5	10	100000
$\overline{w}$	0.0310	0.0357	0.0393	0.0420	0.0440	0.0484	0.0529	0.0565	0.0669
$\overline{\sigma}_{xx}$	6.0875	6.6829	7.1170	7.4309	7.6620	8.1998	9.0737	10.0401	13.1255
$\overline{\sigma}_{yy}$	7.8914	8.6529	9.2061	9.6044	9.8967	10.5760	11.7060	12.9736	17.0149
$\overline{\sigma}_{xy}$	2.5084	2.7557	2.9350	3.0642	3.1592	3.3801	3.7417	4.1419	5.4085
$\overline{\sigma}_{xz}$	1.0975	1.2197	1.3260	1.4140	1.4881	1.6954	1.9694	2.1252	2.3483
M_{xx}	0.0733	0.0842	0.0916	0.0967	0.1003	0.1082	0.1189	0.1290	0.1581
M_{yy}	0.0950	0.1091	0.1186	0.1251	0.1297	0.1399	0.1538	0.1669	0.2049
M_{xy}	0.0316	0.0362	0.0394	0.0416	0.0431	0.0467	0.0516	0.0560	0.0681
M^{f}_{xx}	0.0733	0.0842	0.0916	0.0967	0.1003	0.1082	0.1189	0.1290	0.1581
M^{f}_{yy}	0.0950	0.1091	0.1186	0.1251	0.1297	0.1399	0.1538	0.1669	0.2049
M^{f}_{xy}	0.0316	0.0362	0.0394	0.0416	0.0431	0.0467	0.0516	0.0560	0.0681

In this paper, result obtained for deflection, stresses and moments due to different span ratio for simply supported FGM plates table 1 show the effect of gradation index 'n' for a thick simply supported FGM plates. It is observed from Fig 4 the effect of grading index is more prominent when the value of n is less than 2 for deflection. Fig 5 and Fig 6 represent the through thickness variation of stresses for different values of gradation index 'n'.

Flexural response of FGM plate is presented using shear deformation theory. The effect of span to thickness ratio decreases for $a/h \geq 40$. The effect of gradation index 'n' is prominent for lesser values of 'n' and decreases as 'n' increases. The present results can be used for validation purpose. Present solution mythology is good for obtaining the result and the concentrated load. The same can be extended for other types of concentrated load like sinusoidal varying line load, point load, patch load etc.

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