Fuzzy and Anti-Fuzzy Normal HX Subring of A HX Ring

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Abstract

In this paper, we define the concept of a fuzzy and anti-fuzzy normal HX subring of a HX ring also discuss some related properties of it.

Keywords: HX Ring, Fuzzy HX Ring, Fuzzy Set, Fuzzy Normal HX Subring, Anti-Fuzzy Normal HX Subring

I. INTRODUCTION

In 1965, Zadeh [15] introduced the concept of fuzzy subset of a set X as a function from X into the closed interval 0 and 1 and studied their properties. Fuzzy set theory is a useful tool to describe situations in which the data or imprecise or vague and it is applied to logic , set theory, group theory, ring theory, real analysis, measure theory etc. In 1967, Rosenfeld [13] defined the idea of fuzzy subgroups and gave some of its properties. In 1988, Professor Li Hong Xing [6] proposed the concept of HX ring and derived some of its properties, then Professor Zhong [1,2] gave the structures of HX ring on a class of ring. In this paper, we define the concept of fuzzy normal HX ring and anti-fuzzy normal HX ring of a HX ring by establishing the relation among them and discussed its properties. We also discuss some results based on homomorphism and anti-homomorphism of a fuzzy normal HX ring and anti-fuzzy normal HX ring of a HX ring.

II. PRELIMINARIES

In this section, we site the fundamental definitions that will be used in the sequel. Throughout this paper, R = (R, +, ·) is a Ring, e is the additive identity element of R and xy, we mean x.y.

III. FUZZY NORMAL HX RING

In this section we define the concept of fuzzy normal HX ring and discuss some related results.

A. *Definition*

Let R be a ring. Let μ be a fuzzy set defined on R. Let $\Re \subset 2^R - \{\phi\}$ be a HX ring. A fuzzy HX subring λ^μ of $\Re$ is called a fuzzy normal HX ring on $\Re$ or a fuzzy normal ring induced by μ if the following conditions are satisfied. For all A , B $\in \Re$, $\lambda^\mu(AB) = \lambda^\mu(BA)$, where $\lambda^\mu(A) = \max\{\mu(x) \,/\text{for all } x \in A \subseteq R\}$.

B. *Example*

In Example C [10], λ^μ is a fuzzy HX ring on $\Re$.
For all A , B $\in \Re$, $\lambda^\mu(AB) = \lambda^\mu(BA)$.
Hence, λ^μ is a fuzzy normal HX ring on $\Re$.

C. *Theorem*

If μ is a fuzzy normal subring of a ring R then the fuzzy subset λ^μ is a fuzzy normal HX subring of a HX ring $\Re$.

1) Proof

Let μ be a fuzzy normal subring of R. That is, $\mu(xy) = \mu(yx)$, for all x,y $\in$R.
By Theorem 'D' [10], λ^μ is a fuzzy HX subring on $\Re$.
For all A, B $\in \Re$,

$$\lambda^\mu(AB) = \max\{\mu(xy) \,/\text{ for all } xy \in AB \subseteq R\}$$
$$= \max\{\mu(yx) \,/\text{ for all } xy \in AB \subseteq R\},$$
$$= \lambda^\mu(BA)$$
$$\lambda^\mu(AB) = \lambda^\mu(BA).$$

Hence, λ^μ is a fuzzy normal HX subring of a HX ring $\Re$.

D. Theorem

Let μ and η be any two fuzzy sets defined on R. Let λ^{μ} and γ^{η} be any two fuzzy normal HX subrings of a HX ring $\Re$ then the intersection of two fuzzy normal HX subrings, $\lambda^{\mu} \cap \gamma^{\eta}$ is also a fuzzy normal HX subring of a HX ring $\Re$.

1) Proof

Let μ and η be any two fuzzy sets defined on R.
Let λ^{μ} and γ^{η} be any two fuzzy normal HX subrings of a HX ring $\Re$.
By Theorem 'H' [10], $\lambda^{\mu} \cap \gamma^{\eta}$ is a fuzzy HX subring of a HX ring $\Re$.
Let A, B $\in \Re$.

$$\begin{aligned}(\lambda^{\mu} \cap \gamma^{\eta})(AB) &= \min\{\lambda^{\mu}(AB), \gamma^{\eta}(AB)\}\\ &= \min\{\lambda^{\mu}(BA), \gamma^{\eta}(BA)\},\\ &= (\lambda^{\mu} \cap \gamma^{\eta})(BA).\\ (\lambda^{\mu} \cap \gamma^{\eta})(AB) &= (\lambda^{\mu} \cap \gamma^{\eta})(BA).\end{aligned}$$

Hence, $\lambda^{\mu} \cap \gamma^{\eta}$ is a fuzzy normal HX subring of a HX ring $\Re$.

E. Remark

- The intersection of family of fuzzy normal HX subrings of a HX ring $\Re$ is also fuzzy normal HX subring of $\Re$.
- Let R be a ring. Let μ and η are fuzzy sets of R and $\mu \cap \eta$ is also a fuzzy set of R then $\varphi^{\mu \cap \eta}$ is a fuzzy normal HX subring of $\Re$ induced by $\mu \cap \eta$ of R.

F. Theorem

If λ^{μ}, γ^{η}, $\varphi^{\mu \cap \eta}$ are fuzzy normal HX subrings of a HX ring $\Re$ induced by the fuzzy sets
μ, η , $\mu \cap \eta$ of R then $\varphi^{\mu \cap \eta} = \lambda^{\mu} \cap \gamma^{\eta}$.

1) Proof

By Theorem 'J' [10], it is clear.

G. Theorem

Let μ and η be fuzzy sets of R. Let $\Re \subset 2^{R} - \{\phi\}$ be a HX ring. If λ^{μ} and γ^{η} are any two fuzzy normal HX subrings of $\Re$ then $\lambda^{\mu} \cup \gamma^{\eta}$ is a fuzzy normal HX subring of $\Re$.

1) Proof

Let μ and η be fuzzy sets of R.
Let λ^{μ} and γ^{η} be any two fuzzy normal HX subrings of $\Re$.
By Theorem 'M'[10] , $\lambda^{\mu} \cup \gamma^{\eta}$ is a fuzzy HX subring of $\Re$.
Let A, B $\in \Re$.

$$\begin{aligned}(\lambda^{\mu} \cup \gamma^{\eta})(AB) &= \max\{\lambda^{\mu}(AB), \gamma^{\eta}(AB)\}\\ &= \max\{\lambda^{\mu}(BA), \gamma^{\eta}(BA)\}\\ &= (\lambda^{\mu} \cup \gamma^{\eta})(BA).\\ (\lambda^{\mu} \cup \gamma^{\eta})(AB) &= (\lambda^{\mu} \cup \gamma^{\eta})(BA).\end{aligned}$$

Hence, $\lambda^{\mu} \cup \gamma^{\eta}$ is a fuzzy normal HX subring of $\Re$.

H. Remark

- Union of family of fuzzy normal HX subrings of a HX ring $\Re$ is also fuzzy normal HX subring of $\Re$.
- Let R be a ring. Let μ and η are fuzzy sets of R then $\varphi^{\mu \cup \eta}$ is a fuzzy normal HX subring of $\Re$ induced by the fuzzy set $\mu \cup \eta$ of R.

I. Theorem

Let R be a ring. Let μ and η be fuzzy sets of R. If λ^{μ}, γ^{η} , $\varphi^{\mu \cup \eta}$ are fuzzy normal HX subrings of a HX ring $\Re$ induced by μ, η, $\mu \cup \eta$ of R then $\varphi^{\mu \cup \eta} = \lambda^{\mu} \cup \gamma^{\eta}$.

1) Proof

By Theorem 'O' [10], it is clear.

J. Theorem

Let λ^{μ} and γ^{η} be fuzzy normal HX subrings of HX rings $\Re_1$ and $\Re_2$ respectively then $\lambda^{\mu} \times \gamma^{\eta}$ is also a fuzzy normal HX subring of a HX ring $\Re_1 \times \Re_2$.

1) Proof

Let μ and η be fuzzy sets of R.
Let λ^{μ} and γ^{η} be any two fuzzy normal HX subrings of $\Re_1$ and $\Re_2$ respectively.

By Theorem ‘Q’ [10], $\lambda^{\mu} \times \gamma^{\eta}$ is a fuzzy HX subring of $\Re_1 \times \Re_2$.
Let $(A, B), (C, D) \in \Re_1 \times \Re_2$.

$$(\lambda^{\mu} \times \gamma^{\eta})((A, B)\cdot(C, D)) = (\lambda^{\mu} \times \gamma^{\eta})(AC, BD)$$
$$= \min\{\lambda^{\mu}(AC), \gamma^{\eta}(BD)\}$$
$$= \min\{\lambda^{\mu}(CA), \gamma^{\eta}(DB)\}$$
$$= (\lambda^{\mu} \times \gamma^{\eta})(CA, DB)$$
$$= (\lambda^{\mu} \times \gamma^{\eta})((C, D)\cdot(A, B))$$
$$(\lambda^{\mu} \times \gamma^{\eta})((A, B)\cdot(C, D)) = (\lambda^{\mu} \times \gamma^{\eta})((C, D)\cdot(A, B)).$$

Hence, $\lambda^{\mu} \times \gamma^{\eta}$ is a fuzzy normal HX subring of a HX ring $\Re_1 \times \Re_2$.

K. Theorem

Let λ^{μ} and γ^{η} be fuzzy subsets of the HX rings $\Re_1$ and $\Re_2$ respectively, such that $\lambda^{\mu}(A) \leq \gamma^{\eta}(Q^1)$ for all $A \in \Re_1$, Q^1 being the identity element of $\Re_2$. If $(\lambda^{\mu} \times \gamma^{\eta})$ is a fuzzy normal HX subring of $\Re_1 \times \Re_2$ then λ^{μ} is a fuzzy normal HX subring of $\Re_1$.

1) Proof

Let $\lambda^{\mu} \times \gamma^{\eta}$ be a fuzzy normal HX subring of $\Re_1 \times \Re_2$.
By Theorem ‘S’[10], λ^{μ} is a fuzzy HX subring of $\Re_1$.
Let $A, B \in \Re_1$.
Given,

$$\lambda^{\mu}(A) \leq \gamma^{\eta}(Q^1) \text{ for all } A \in \Re_1.$$
$$\lambda^{\mu}(AB) = \min\{\lambda^{\mu}(AB), \gamma^{\eta}(Q^1Q^1)\}$$
$$= (\lambda^{\mu} \times \gamma^{\eta})((A, Q^1)\cdot(B, Q^1))$$
$$= (\lambda^{\mu} \times \gamma^{\eta})((B, Q^1)\cdot(A, Q^1))$$
$$= \min\{\lambda^{\mu}(BA), \gamma^{\eta}(Q^1Q^1)\}$$
$$= \lambda^{\mu}(BA).$$
$$\lambda^{\mu}(AB) = \lambda^{\mu}(BA).$$

Hence, λ^{μ} is a fuzzy normal HX subring of $\Re_1$.

L. Theorem

Let λ^{μ} and γ^{η} be fuzzy subsets of the HX rings $\Re_1$ and $\Re_2$ respectively, such that $\gamma^{\eta}(A) \leq \lambda^{\mu}(Q)$ for all $A \in \Re_2$, Q being the identity element of $\Re_1$. If $\lambda^{\mu} \times \gamma^{\eta}$ is a fuzzy normal HX sub ring of $\Re_1 \times \Re_2$ then γ^{η} is a fuzzy normal HX subring of $\Re_2$.

1) Proof

Let $\lambda^{\mu} \times \gamma^{\eta}$ be a fuzzy normal HX subring of $\Re_1 \times \Re_2$.
By Theorem ‘T’ [10], γ^{η} is a fuzzy HX subring of $\Re_2$.
Let $A, B \in \Re_2$.
Given,

$$\gamma^{\eta}(A) \leq \lambda^{\mu}(Q) \text{ for all } A \in \Re_2.$$
$$\gamma^{\eta}(AB) = \min\{\lambda^{\mu}(QQ), \gamma^{\eta}(AB)\}$$
$$= (\lambda^{\mu} \times \gamma^{\eta})((Q, A)\cdot(Q, B))$$
$$= (\lambda^{\mu} \times \gamma^{\eta})((Q, B)\cdot(Q, A))$$
$$= \min\{\lambda^{\mu}(QQ), \gamma^{\eta}(BA)\}$$
$$= \gamma^{\eta}(BA)$$
$$\gamma^{\eta}(AB) = \gamma^{\eta}(BA)$$

Hence, γ^{η} is a fuzzy normal HX subring of $\Re_2$.

IV. HOMOMORPHISM AND ANTI-HOMOMORPHISM OF A FUZZY NORMAL HX SUBRING OF A HX RING

In this section, we introduce the concept of an image, pre-image of fuzzy subset of a HX ring and discuss the properties of homomorphic and anti homomorphic images and pre images of fuzzy normal HX subring of a HX ring $\Re$.

A. Theorem

Let $\Re_1$ and $\Re_2$ be any two HX rings on the rings R_1 and R_2 respectively. Let $f : \Re_1 \rightarrow \Re_2$ be a homomorphism onto HX rings. Let λ^{μ} be a fuzzy normal HX subring of $\Re_1$ then $f(\lambda^{\mu})$ is a fuzzy normal HX subring of $\Re_2$, if λ^{μ} has a supremum property and λ^{μ} is f-invariant.

1) Proof

Let μ be a fuzzy subset of R_1 and λ^{μ} is a fuzzy normal HX subring of $\Re_1$.
By Theorem 3.4[12], $f(\lambda^{\mu})$ is a fuzzy HX subring of $\Re_2$.

There exist $X,Y \in \Re_1$ such that $f(X), f(Y) \in \Re_2$,

$$(f(\lambda^{\mu}))(f(X)\,f(Y)) = (f(\lambda^{\mu}))(f(XY)),$$
$$= \lambda^{\mu}(XY)$$
$$= \lambda^{\mu}(YX)$$
$$= (f(\lambda^{\mu}))(f(YX))$$
$$= (f(\lambda^{\mu}))(f(Y)f(X))$$
$$(f(\lambda^{\mu}))(f(X)f(Y)) = (f(\lambda^{\mu}))(f(Y)f(X)).$$

Hence, $f(\lambda^{\mu})$ is a fuzzy normal HX subring of $\Re_2$.

B. Theorem

Let $\Re_1$ and $\Re_2$ be any two HX rings on R_1 and R_2 respectively. Let $f: \Re_1 \rightarrow \Re_2$ be a homomorphism on HX rings. Let η^{α} be a fuzzy normal HX subring of $\Re_2$ then $f^{-1}(\eta^{\alpha})$ is a fuzzy normal HX subring of $\Re_1$.

1) Proof

Let α be a fuzzy subset of R_2 and η^{α} be a fuzzy normal HX subring of $\Re_2$.
By Theorem 3.5 [12], $f^{-1}(\eta^{\alpha})$ is a fuzzy HX subring of $\Re_1$.
For any $X,Y \in \Re_1$, $f(X), f(Y) \in \Re_2$,

$$(f^{-1}(\eta^{\alpha}))(XY) = \eta^{\alpha}(f(XY))$$
$$= \eta^{\alpha}(f(X)\,f(Y))$$
$$= \eta^{\alpha}(f(Y)\,f(X))$$
$$= \eta^{\alpha}(f(YX))$$
$$= (f^{-1}(\eta^{\alpha}))(YX)$$
$$(f^{-1}(\eta^{\alpha}))(XY) = (f^{-1}(\eta^{\alpha}))(YX)$$

Hence, $f^{-1}(\eta^{\alpha})$ is a fuzzy normal HX subring of $\Re_1$.

C. Theorem

Let $\Re_1$ and $\Re_2$ be any two HX rings on R_1 and R_2 respectively. Let $f: \Re_1 \rightarrow \Re_2$ be an anti homomorphism onto HX rings. Let λ^{μ} be a fuzzy normal HX subring of $\Re_1$, then $f(\lambda^{\mu})$ is a fuzzy normal HX subring of $\Re_2$, if λ^{μ} has a supremum property and λ^{μ} is f-invariant.

1) Proof

Let μ be a fuzzy subset of R_1 and λ^{μ} is a fuzzy normal HX subring of $\Re_1$.
By Theorem 3.6[12], $f(\lambda^{\mu})$ is a fuzzy HX subring of $\Re_2$.
There exist $X, Y \in \Re_1$ such that $f(X), f(Y) \in \Re_2$,

$$(f(\lambda^{\mu}))(f(X)\,f(Y)) = (f(\lambda^{\mu}))(f(YX)),$$
$$= \lambda^{\mu}(YX)$$
$$= \lambda^{\mu}(XY)$$
$$= (f(\lambda^{\mu}))(f(XY))$$
$$= (f(\lambda^{\mu}))(f(Y)f(X))$$
$$(f(\lambda^{\mu}))(f(X)f(Y)) = (f(\lambda^{\mu}))(f(Y)f(X)).$$

Hence, $f(\lambda^{\mu})$ is a fuzzy normal HX subring of $\Re_2$.

D. Theorem

Let $\Re_1$ and $\Re_2$ be any two HX rings on R_1 and R_2 respectively. Let $f: \Re_1 \rightarrow \Re_2$ be an anti homomorphism on HX rings. Let η^{α} be a fuzzy normal HX subring of $\Re_2$ then $f^{-1}(\eta^{\alpha})$ is a fuzzy normal HX subring of $\Re_1$.

1) Proof

Let α be a fuzzy subset of R_2 and η^{α} be a fuzzy normal HX subring of $\Re_2$.
By Theorem 3.7 [12], $f^{-1}(\eta^{\alpha})$ is a fuzzy HX subring of $\Re_1$.

For any $X,Y \in \Re_1$, $f(X), f(Y) \in \Re_2$,

$$(f^{-1}(\eta^{\alpha}))(XY) = \eta^{\alpha}(f(XY))$$
$$= \eta^{\alpha}(f(Y)\,f(X))$$
$$= \eta^{\alpha}(f(X)\,f(Y))$$
$$= \eta^{\alpha}(f(YX))$$
$$= (f^{-1}(\eta^{\alpha}))(YX)$$
$$(f^{-1}(\eta^{\alpha}))(XY) = (f^{-1}(\eta^{\alpha}))(YX)$$

Hence, $f^{-1}(\eta^{\alpha})$ is a fuzzy normal HX subring of $\Re_1$.

V. ANTI-FUZZY NORMAL HX RING

In this section we define the concept of anti-fuzzy normal HX ring and discuss some related results.

A. Definition

Let R be a ring. Let μ be a fuzzy set defined on R. Let $\Re \subset 2^R - \{\phi\}$ be a HX ring. A fuzzy HX subring λ_μ of $\Re$ is called an anti-fuzzy normal HX ring on $\Re$ or an anti-fuzzy ring induced by μ if the following conditions are satisfied. For all A , B $\in \Re$,

$$\lambda_\mu(AB) = \lambda_\mu(BA), \text{ where } \lambda_\mu(A) = \min\{\mu(x)/\text{for all } x \in A \subseteq R\}.$$

B. Theorem

If μ is an anti-fuzzy normal subring of a ring R then the fuzzy subset λ_μ is an anti-fuzzy normal HX subring of a HX ring $\Re$.

1) Proof

Let μ be an anti-fuzzy normal subring of R. That is, μ(xy) = μ(yx), for all x,y $\in$R.
By Theorem 3.6 [11] , λ_μ is an anti-fuzzy normal HX subring of a HX ring $\Re$.
For all A, B $\in \Re$,

$$\lambda_\mu(AB) = \min\{\mu(xy) \;/\text{ for all } xy \in AB \subseteq R\}$$
$$= \min\{\mu(yx) \;/\text{ for all } xy \in AB \subseteq R\}$$
$$= \lambda_\mu(BA)$$
$$\lambda_\mu(AB) = \lambda_\mu(BA).$$

Hence, λ_μ is an anti-fuzzy normal HX subring of a HX ring $\Re$.

C. Theorem

Let μ and η be any two fuzzy sets of R. Let $\Re \subset 2^R - \{\phi\}$ be a HX ring. Let λ_μ and γ_η be any two fuzzy normal HX subrings of a HX ring $\Re$ then the intersection of two anti-fuzzy normal HX subrings, $\lambda_\mu \cap \gamma_\eta$ is also an anti-fuzzy normal HX subring of a HX ring $\Re$.

1) Proof

It is clear.

D. Remark

- The intersection of family of anti-fuzzy normal HX subrings of a HX ring $\Re$ is also an anti-fuzzy HX subring of $\Re$.
- Let R be a ring. Let μ and η are fuzzy sets of R and μ ∩ η is also a fuzzy subset of R then φμ ∩ η is an anti-fuzzy normal HX subring of $\Re$ induced by μ ∩ η of R.

E. Theorem

Let R be a ring. Let μ and η are fuzzy sets of R. If λ_μ , γ_η , $\varphi_{\mu\cap\eta}$ are anti-fuzzy normal HX subrings of a HX ring $\Re$ induced by μ , η , μ ∩ η of R then $\varphi_{\mu\cap\eta} = \lambda_\mu \cap \gamma_\eta$.

1) Proof

It is clear.

F. Theorem

Let μ and η are any two fuzzy sets of R. Let $\Re \subset 2^R - \{\phi\}$ be a HX ring. If λ_μ and γ_η be any two anti-fuzzy normal HX subrings of $\Re$ then $\lambda_\mu \cup \gamma_\eta$ is also an anti-fuzzy normal HX subring of $\Re$.

1) Proof

It is clear.

G. Remark

- Union of family of anti-fuzzy normal HX subrings of a HX ring $\Re$ is also an anti-fuzzy normal HX subring of $\Re$.
- Let R be a ring. Let μ and η be fuzzy sets of R then $\varphi_{\mu\cup\eta}$ is an anti-fuzzy normal HX subring of $\Re$ induced by μ ∪ η of R.

H. Theorem

Let R be a ring. Let μ and η be fuzzy sets of R. If λ_μ , γ_η , $\varphi_{\mu\cup\eta}$ are anti-fuzzy normal HX subrings of a HX ring $\Re$ induced by μ , η , μ ∪ η of R then $\varphi_{\mu\cup\eta} = \lambda_\mu \cup \gamma_\eta$.

1) Proof

It is clear.

I. *Theorem*

Let λ_μ and γ_η be anti-fuzzy normal HX subrings of HX rings $\mathfrak{R}_1$ and $\mathfrak{R}_2$ respectively then the cartesian anti-product of λ_μ and γ_η denoted as $\lambda_\mu \times \gamma_\eta$ is also an anti-fuzzy normal HX subring of $\mathfrak{R}_1 \times \mathfrak{R}_2$.

1) Proof

Let μ and η be fuzzy sets of R.
Let λ_μ and γ_η be anti-fuzzy normal HX subrings of HX rings $\mathfrak{R}_1$ and $\mathfrak{R}_2$ respectively.
By Theorem 4.2[11], $\lambda_\mu \times \gamma_\eta$ is an anti-fuzzy HX subring of $\mathfrak{R}_1 \times \mathfrak{R}_2$.

$$\text{Let } (A, B), (C, D) \in \mathfrak{R}_1 \times \mathfrak{R}_2.$$
$$(\lambda_\mu \times \gamma_\eta)((A, B)\cdot(C, D)) = (\lambda_\mu \times \gamma_\eta)(AC, BD)$$
$$= \max\{\lambda_\mu(AC), \gamma_\eta(BD)\}$$
$$= \max\{\lambda_\mu(CA), \gamma_\eta(DB)\}$$
$$= (\lambda_\mu \times \gamma_\eta)(CA, DB)$$
$$= (\lambda_\mu \times \gamma_\eta)((C, D)\cdot(A, B))$$
$$(\lambda_\mu \times \gamma_\eta)((A, B)\cdot(C, D)) = (\lambda_\mu \times \gamma_\eta)((C, D)\cdot(A, B)).$$

Hence, $\lambda_\mu \times \gamma_\eta$ is also an anti-fuzzy normal HX subring of a HX ring $\mathfrak{R}$.

J. *Theorem*

Let λ_μ and γ_η be fuzzy subsets of the HX rings $\mathfrak{R}_1$ and $\mathfrak{R}_2$ respectively, such that $\lambda_\mu(A) \geq \gamma_\eta(Q^1)$ for all $A \in \mathfrak{R}_1$, Q^1 being the identity element of $\mathfrak{R}_2$ .If $(\lambda_\mu \times \gamma_\eta)$ is an anti-fuzzy normal HX subring of $\mathfrak{R}_1 \times \mathfrak{R}_2$ then λ_μ is an anti-fuzzy normal HX subring of $\mathfrak{R}_1$.

1) Proof

Let $(\lambda_\mu \times \gamma_\eta)$ is an anti-fuzzy normal HX subring of $\mathfrak{R}_1 \times \mathfrak{R}_2$.
By Theorem 4.3[11], λ_μ is an anti-fuzzy HX subring of $\mathfrak{R}_1$.
Given,

$$\lambda_\mu(A) \geq \gamma_\eta(Q^1) \text{ for all } A \in \mathfrak{R}_1.$$
$$\lambda_\mu(AB) = \max\{\lambda_\mu(AB), \gamma_\eta(Q^1Q^1)\}$$
$$= (\lambda_\mu \times \gamma_\eta)((A, Q^1)\cdot(B, Q^1))$$
$$= (\lambda_\mu \times \gamma_\eta)((B, Q^1)\cdot(A, Q^1))$$
$$= \max\{\lambda_\mu(BA), \gamma_\eta(Q^1Q^1)\}$$
$$= \lambda_\mu(BA)$$
$$\lambda_\mu(AB) = \lambda_\mu(BA)$$

Hence, λ_μ is an anti-fuzzy normal HX subring of $\mathfrak{R}_1$.

K. *Theorem*

Let λ_μ and γ_η be fuzzy subsets of the HX rings $\mathfrak{R}_1$ and $\mathfrak{R}_2$ respectively , such that $\gamma_\eta(A) \geq \lambda_\mu(Q)$ for all $A \in \mathfrak{R}_2$, Q being the identity element of $\mathfrak{R}_1$. If $\lambda_\mu \times \gamma_\eta$ is an anti-fuzzy normal HX sub ring of $\mathfrak{R}_1 \times \mathfrak{R}_2$ then γ_η is an anti-fuzzy normal HX subring of $\mathfrak{R}_2$.

1) Proof

Let $\lambda_\mu \times \gamma_\eta$ be an anti-fuzzy normal HX subring of $\mathfrak{R}_1 \times \mathfrak{R}_2$.
By Theorem 4.4[11], γ_η is an anti-fuzzy HX subring of $\mathfrak{R}_2$.
Let A, B $\in \mathfrak{R}_2$.
Given

$$\gamma_\eta(A) \geq \lambda_\mu(Q) \text{ for all } A \in \mathfrak{R}_2$$
$$\gamma_\eta(AB) = \max\{\lambda_\mu(QQ), \gamma_\eta(AB)\}$$
$$= (\lambda_\mu \times \gamma_\eta)((Q, A)\cdot(Q, B))$$
$$= (\lambda_\mu \times \gamma_\eta)((Q, B)\cdot(Q, A))$$
$$= \max\{\lambda_\mu(QQ), \gamma_\eta(BA)\}$$
$$\gamma_\eta(AB) = \gamma_\eta(BA)\}.$$

Hence, γ_η is an anti-fuzzy normal HX subring of $\mathfrak{R}_2$.

L. *Theorem*

Let μ be a fuzzy set defined on R. Let λ^μ be a fuzzy normal HX subring of $\mathfrak{R}$ if and only if $(\lambda^\mu)^c$ is an anti-fuzzy normal HX subring of $\mathfrak{R}$.

1) Proof

It is clear.

VI. HOMOMORPHISM AND ANTI-HOMOMORPHISM OF AN ANTI-FUZZY NORMAL HX SUBRING OF A HX RING

In this section, we introduce the concept of an anti-image, anti-pre-image of an anti-fuzzy normal HX subring of a HX ring and discussed its properties under homomorphism and anti-homomorphism.

A. *Theorem*

Let $\Re_1$ and $\Re_2$ be any two HX rings on the rings R_1 and R_2 respectively. Let f: $\Re_1 \rightarrow \Re_2$ be a homomorphism onto HX rings. Let λ_μ be an anti-fuzzy normal HX subring of $\Re_1$ then f (λ_μ) is an anti-fuzzy normal HX subring of $\Re_2$, if λ_μ has an infimum property and λ_μ is f-invariant.

1) *Proof*

Let μ be a fuzzy subset of R_1 and λ_μ is an anti-fuzzy HX subring of $\Re_1$.

There exist X, Y$\in \Re_1$ such that f(X) , f(Y) $\in \Re_2$,

$$\begin{aligned}(f(\lambda_\mu))(f(X)-f(Y)) &= (f(\lambda_\mu))(f(X-Y)),\\ &= \lambda_\mu(X-Y)\\ &\leq \max\{\lambda_\mu(X), \lambda_\mu(Y)\}\\ &= \max\{(f(\lambda_\mu))(f(X)), (f(\lambda_\mu))(f(Y))\}\end{aligned}$$

$$(f(\lambda_\mu))(f(X)-f(Y)) \leq \max\{(f(\lambda_\mu))(f(X)), (f(\lambda_\mu))(f(Y))\}.$$

$$\begin{aligned}((f(\lambda_\mu))(f(X)f(Y)) &= (f(\lambda_\mu))(f(XY)),\\ &= \lambda_\mu(XY)\\ &\leq \max\{\lambda_\mu(X), \lambda_\mu(Y)\}\\ &= \max\{(f(\lambda_\mu))(f(X)), (f(\lambda_\mu))(f(Y))\}\end{aligned}$$

$$(f(\lambda_\mu))(f(X)f(Y)) \leq \max\{(f(\lambda_\mu))(f(X), (f(\lambda_\mu))(f(Y))\}.$$

Hence, f (λ_μ) is an anti-fuzzy HX subring of $\Re_2$.

Also

$$\begin{aligned}(f(\lambda_\mu))(f(X)f(Y)) &= (f(\lambda_\mu))(f(XY)),\\ &= \lambda_\mu(XY)\\ &= \lambda_\mu(YX)\\ &= (f(\lambda_\mu))(f(YX))\end{aligned}$$

$$(f(\lambda_\mu))(f(X)f(Y)) = (f(\lambda_\mu))(f(Y)f(X)).$$

Hence, f (λ_μ) is an anti-fuzzy normal HX subring of $\Re_2$.

B. *Theorem*

Let $\Re_1$ and $\Re_2$ be any two HX rings on R_1 and R_2 respectively. Let f: $\Re_1 \rightarrow \Re_2$ be a homomorphism on HX rings. Let η_α be an anti-fuzzy normal HX subring of $\Re_2$ then $f^{-1}(\eta_\alpha)$ is an anti-fuzzy normal HX subring of $\Re_1$.

1) *Proof*

Let α be a fuzzy subset of R_2 and η_α be an anti-fuzzy HX subring of $\Re_2$.

For any X,Y $\in \Re_1$, f(X) , f(Y) $\in \Re_2$,

$$\begin{aligned}(f^{-1}(\eta_\alpha))(X-Y) &= \eta_\alpha(f(X-Y))\\ &= \eta_\alpha(f(X)-f(Y)),\\ &\leq \max\{\eta_\alpha(f(X)), \eta_\alpha(f(Y))\}\\ &= \max\{(f^{-1}(\eta_\alpha))(X), (f^{-1}(\eta_\alpha))(Y)\}\end{aligned}$$

$$(f^{-1}(\eta_\alpha))(X-Y) \leq \max\{(f^{-1}(\eta_\alpha))(X), (f^{-1}(\eta_\alpha))(Y)\}.$$

$$\begin{aligned}(f^{-1}(\eta_\alpha))(XY) &= \eta_\alpha(f(XY))\\ &= \eta_\alpha(f(X)f(Y))\\ &\leq \max\{\eta_\alpha(f(X)), \eta_\alpha(f(Y))\}\\ &= \max\{(f^{-1}(\eta_\alpha))(X), (f^{-1}(\eta_\alpha))(Y)\}\end{aligned}$$

$$(f^{-1}(\eta_\alpha))(XY) \leq \max\{(f^{-1}(\eta_\alpha))(X), (f^{-1}(\eta_\alpha))(Y)\}.$$

Hence, $f^{-1}(\eta_\alpha)$ is an anti-fuzzy HX subring of $\Re_1$.

Also,

$$\begin{aligned}(f^{-1}(\eta_\alpha))(XY) &= \eta_\alpha(f(XY))\\ &= \eta_\alpha(f(X)f(Y))\\ &= \eta_\alpha(f(Y)f(X))\\ &= \eta_\alpha(f(YX))\\ &= f^{-1}(\eta_\alpha))(YX)\end{aligned}$$

$$(f^{-1}(\eta_\alpha))(XY) = f^{-1}(\eta_\alpha))(YX).$$

Hence, $f^{-1}(\eta_\alpha)$ is an anti-fuzzy normal HX subring of $\Re_1$.

C. Theorem

Let $\Re_1$ and $\Re_2$ be any two HX rings on the rings R_1 and R_2 respectively. Let f: $\Re_1 \to \Re_2$ be an anti-homomorphism onto HX rings. Let λ_μ be an anti-fuzzy normal HX subring of $\Re_1$ then f (λ_μ) is an anti-fuzzy normal HX subring of $\Re_2$, if λ_μ has an infimum property and λ_μ is f-invariant.

1) Proof

Let μ be a fuzzy subset of R_1 and λ_μ is an anti-fuzzy HX subring of $\Re_1$, then
There exist X, Y $\in \Re_1$ such that f(X), f (Y) $\in \Re_2$

$$(f(\lambda_\mu))(f(X)-f(Y)) = (f(\lambda_\mu))(f(Y-X)),$$
$$= \lambda_\mu(Y-X)$$
$$\leq \max\{\lambda_\mu(Y), \lambda_\mu(X)\}$$
$$= \max\{(f(\lambda_\mu))(f(Y)), (f(\lambda_\mu))(f(X))\}$$
$$(f(\lambda_\mu))(f(X)-f(Y)) \leq \max\{(f(\lambda_\mu))(f(X)), (f(\lambda_\mu))(f(Y))\}.$$
$$(f(\lambda_\mu))(f(X)\,f(Y)) = (f(\lambda_\mu))(f(YX)),$$
$$= \lambda_\mu(YX)$$
$$\leq \max\{\lambda_\mu(Y), \lambda_\mu(X)\}$$
$$= \max\{(f(\lambda_\mu))(f(Y)), (f(\lambda_\mu))(f(X))\}$$
$$(f(\lambda_\mu))(f(X)f(Y)) \leq \max\{(f(\lambda_\mu))(f(X)), (f(\lambda_\mu))(f(Y))\}.$$

Hence, f (λ_μ) is an anti-fuzzy HX subring of $\Re_2$.
Also

$$(f(\lambda_\mu))(f(X)\,f(Y)) = (f(\lambda_\mu))(f(YX)),$$
$$= \lambda_\mu(YX)$$
$$= \lambda_\mu(XY)$$
$$= (f(\lambda_\mu))(f(XY))$$
$$(f(\lambda_\mu))(f(X)f(Y)) = (f(\lambda_\mu))(f(Y)f(X)).$$

Hence, f (λ_μ) is an anti-fuzzy normal HX subring of $\Re_2$.

D. Theorem

Let $\Re_1$ and $\Re_2$ be any two HX rings on R_1 and R_2 respectively. Let f: $\Re_1 \to \Re_2$ be an anti-homomorphism on HX rings. Let η_α be an anti-fuzzy normal HX subring of $\Re_2$ then $f^{-1}(\eta_\alpha)$ is an anti-fuzzy normal HX subring of $\Re_1$.

1) Proof

Let α be a fuzzy subset of R_2 and η_α be an anti-fuzzy HX subring of $\Re_2$.
For any X,Y $\in \Re_1$, then f (X) , f (Y) $\in \Re_2$

$$(f^{-1}(\eta_\alpha))(X-Y) = \eta_\alpha(f(Y-X))$$
$$= \eta_\alpha(f(Y)-f(X)),$$
$$\leq \max\{\eta_\alpha(f(Y)), \eta_\alpha(f(X))\}$$
$$\leq \max\{(f^{-1}(\eta_\alpha))(Y), (f^{-1}(\eta_\alpha))(X)\}$$
$$(f^{-1}(\eta_\alpha))(X-Y) \leq \max\{(f^{-1}(\eta_\alpha))(X), (f^{-1}(\eta_\alpha))(Y)\}.$$
$$(f^{-1}(\eta_\alpha))(XY) = \eta_\alpha(f(YX))$$
$$= \eta_\alpha(f(Y)\,f(X))$$
$$\leq \max\{\eta_\alpha(f(Y)), \eta_\alpha(f(X))\}$$
$$\leq \min\{(f^{-1}(\eta_\alpha))(Y), (f^{-1}(\eta_\alpha))(X)\}$$
$$(f^{-1}(\eta_\alpha))(XY) \leq \max\{(f^{-1}(\eta_\alpha))(Y), (f^{-1}(\eta_\alpha))(X)\}.$$

Therefore, $f^{-1}(\eta_\alpha)$ is an anti-fuzzy HX subring of $\Re_1$.
Also,

$$(f^{-1}(\eta_\alpha))(XY) = \eta_\alpha(f(XY))$$
$$= \eta_\alpha(f(Y)\,f(X))$$
$$= \eta_\alpha(f(X)\,f(Y))$$
$$= \eta_\alpha(f(YX))$$
$$= f^{-1}(\eta_\alpha))(YX)$$
$$(f^{-1}(\eta_\alpha))(XY) = f^{-1}(\eta_\alpha))(YX).$$

Hence, $f^{-1}(\eta_\alpha)$ is an anti-fuzzy normal HX subring of $\Re_1$.

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