A Generalized Metric Space and Related Fixed Point Theorems

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Abstract

We introduce a new concept of generalized metric spaces and extend some well-known related fixed point theorems including Banach contraction principle, Ćirić's fixed point theorem, a fixed point result due to Ran and Reurings, and a fixed point result due to Nieto and Rodríguez-López. This new concept of generalized metric spaces and extend some well-known related fixed point theorems recover various topological spaces including standard metric spaces, b-metric spaces, dislocated metric spaces, and modular spaces.

Keywords: metric spaces, b-metric spaces, dislocated metric spaces, modular space, fixed point partial order

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I. Introduction

The concept of standard metric spaces is a fundamental tool in topology, functional analysis and nonlinear analysis. This structure has attracted a considerable attention from mathematicians because of the development of the fixed point theory in standard metric spaces.

In recent years, several generalizations of standard metric spaces have appeared. In 1993, Czerwik [1] introduced the concept of a *b*-metric space. Since then, several works have dealt with fixed point theory in such spaces; see [2,3, 4, 5, 6, 7] and references therein. In 2000, Hitzler and Seda [8] introduced the notion of dislocated metric spaces in which self-distance of a point need not be equal to zero. Such spaces play a very important role in topology and logical programming. For fixed point theory in dislocated metric spaces, see [9, 10, 11, 12] and references therein. The theory of modular spaces was initiated by Nakano [13] in connection with the theory of order spaces and was redefined and generalized by Musielak and Orlicz [14]. By defining a norm, particular Banach spaces of functions can be considered. Metric fixed theory for these Banach spaces of functions has been widely studied. Even though a metric is not defined, many problems in fixed point theory can be reformulated in modular spaces [15, 16, 17,18, 19, 20,24,25].

In this work, we present a new generalized metric spaces and extend some well-known related fixed point theorems that recovers a large class of topological spaces including standard metric spaces, *b*-metric spaces, dislocated metric spaces, and modular spaces. In such spaces, we establish new versions of some known fixed point theorems in standard metric spaces including Banach contraction principle, Ćirić's fixed point theorem, a fixed point result due to Ran and Reurings, and a fixed point result due to Nieto and Rodíguez-López.

II. A Generalized Metric Space

Let X be a nonempty set and $D:X\times X\rightarrow[0,+\infty]$ be a given mapping. For every $x\in X$, let us define the set

$$C(D,X,x)=\{\{x_n\}\subset X:\lim_{n\rightarrow\infty}D(x_n,x)=0\}.$$

A. General definition

1) Definition 2.1

We say that D is a generalized metric on X if it satisfies the following conditions:

1) *(D1):* for every $(x,y)\in X\times X$, we have $D(x,y)=0\Longrightarrow x=y$;
2) *(D2):* for every $(x,y)\in X\times X$, we have $D(x,y)=D(y,x)$;
3) *(D3):* there exists $K>0$ such that

if $(x,y)\in X\times X,\{x_n\}\in K(D,X,x)$, then $D(x,y)\leq K\limsup_{n\rightarrow\infty}D(x_n,y)$.

In this case, we say the pair (X,D) is a generalized metric space.

B. Topological concepts

1) Definition 2.3

Let (X,D) be a generalized metric space. Let $\{x_n\}$ be a sequence in X and $x\in X$. We say that $\{x_n\}$ D-converges to x if $\{x_n\}\in K(D,X,x)$.

2) *Proposition 2.4*

Let (X,D) *be a generalized metric space*. *Let* {xn} *be a sequence in X and* (x,y)∈X×X. *If* {xn} D-*converges to x and* {xn} D-*converges to y, then* x=y.

3) *Definition 2.5*

Let (X,D) be a generalized metric space. Let {xn} be a sequence in *X*. We say that {xn} is a D-Cauchy sequence if limm,n→∞D(xn,xn+m)=0.

4) *Definition 2.6*

Let (X,D) be a generalized metric space. It is said to be D-complete if every Cauchy sequence in *X* is convergent to some element in *X*.

C. *Examples*

In this part of the paper, we will see that this new concept of generalized metric spaces recovers a large class of existing metrics in the literature.

1) *Standard Metric Spaces*

Recall that a standard metric on a nonempty set *X* is a mapping d:X×X→[0,+∞) satisfying the following conditions:

1) (d_1): for every (x,y)∈X×X, we have d(x,y)=0⟺x=y;
2) (d_2): for every (x,y)∈X×X, we have d(x,y)=d(y,x);
3) (d_3): for every (x,y,z)∈X×X×X, we have d(x,y)≤d(x,z)+d(z,y).

2) *b-Metric spaces*

In 1993, Czerwik [1] introduced the concept of *b*-metric spaces by relaxing the triangle inequality as follows.

3) *Definition 2.7*

Let *X* be a nonempty set and d:X×X→[0,+∞) be a given mapping. We say that *d* is a *b*-metric on *X* if it satisfies the following conditions:

1) (b_1): for every (x,y)∈X×X, we have d(x,y)=0⟺x=y;
2) (b_2): for every (x,y)∈X×X, we have d(x,y)=d(y,x);
3) (b_3): there exists t≥1 such that, for every (x,y,z)∈X×X×X, we have d(x,y)≤t[d(x,z)+d(z,y)].

In this case,(X,d) is said to be a *b*-metric space.

The concept of convergence in such spaces is similar to that of standard metric spaces.

4) *Proposition 2.8*

Any b-metric on X is a generalized metric on X.

5) *Hitzler-Seda metric spaces*

Hitzler and Seda [8, 26] introduced the notion of dislocated metric spaces as follows.

6) *Definition 2.9*

Let *X* be a nonempty set and d:X×X→[0,+∞) be a given mapping. We say that *d* is a dislocated metric on*X* if it satisfies the following conditions:

1) (HS_1): for every (x,y)∈X×X, we have d(x,y)=0⟹x=y;
2) (HS_2): for every (x,y)∈X×X, we have d(x,y)=d(y,x);
3) (HS_3): for every (x,y,z)∈X×X×X, we have d(x,y)≤d(x,z)+d(z,y).

In this case,(X,d) is said to be a dislocated metric space.

The concept of convergence in such spaces is similar to that of standard metric spaces.

7) *Proposition 2.10*

Any dislocated metric on X is a generalized metric on X.

8) *Modular spaces*

Let us recall briefly some basic concepts of modular spaces. For more details of modular spaces, the reader is advised to consult [19], and the references therein.

9) *Definition 2.11*

Let *X* be a linear space over R. A functional q:X→[0,+∞] is said to be modular if the following conditions hold:

1) *(*q1q1*):* for every x∈X, we have q(x)=0⟺x=0;
2) *(*q2q2*):* for every x∈X, we have q(−x)=q(x);
3) *(*q3q3*):* for every (x,y)∈X×X, we have q(αx+βy)≤q(x)+q(y),whenever α,β≥0 and α+β=1.

10) *Definition 2.12*

If *q* is a modular on *X*, then the set Xq={x∈X:limλ→0q(λx)=0}is called a modular space.

The concept of convergence in such spaces is defined as follows.

11) *Definition 2.13*

Let (X,q) be a modular space.

1) A sequence {xn}n∈N⊂Xq is said to be *q*-convergent to x∈Xq if limn→∞q(xn−x)=0.
2) A sequence {xn}n∈N⊂Xq is said to be *q*-Cauchy if limn,m→∞q(xn−xn+m)=0.
3) Xq is said to be *q*-complete if any *q*-Cauchy sequence is *q*-convergent.

12) *Definition 2.14*

The modular q has the Fatou property if, for every $y \in X_q$, we have $q(x-y) \leq \liminf_{n\to\infty} q(x_n-y)$, whenever $\{x_n\}_{n\in N} \subset X_q$ is q-convergent to $x \in X_q$.

Let (X,q) be a modular space. Define the mapping $D_q: X_q \times X_q \to [0,+\infty]$ by $D_q(x,y)=q(x-y)$, for every $(x,y) \in X \times X$.

We have the following result.

13) *Proposition 2.15*

If q has the Fatou property, then D_q is a generalized metric on X_q.

14) *Proposition 2.16*

Let q be a modular on X having the Fatou property. Then

1) $\{x_n\} \subset X_q$ is q-convergent to $x \in X_q$ if and only if $\{x_n\}$ is D_q-convergent to x;
2) $\{x_n\} \subset X_q$ is q-Cauchy if and only if $\{x_n\}$ is D_q-Cauchy;
3) (X_q,q) is q-complete if and only if (X_q,D_q) is D_q-complete.

The Banach contraction principle in a generalized metric space

In this section, we present an extension of the Banach contraction principle to the setting of generalized metric spaces introduced previously.

Let (X,D) be a generalized metric space and $f:X \to X$ be a mapping.

15) *Definition 3.1*

Let $t \in (0,1)$. We say that f is a k-contraction if $D(f(x),f(y)) \leq kD(x,y)$, for every $(x,y) \in X \times X$.

First, we have the following observation.

16) *Proposition 3.2*

Suppose that f is a k-contraction for some $t \in (0,1)$. Then any fixed point $\omega \in X$ if f satisfies $D(\omega,\omega)<\infty \Longrightarrow D(\omega,\omega)=0$.

17) *Theorem 3.3*

Suppose that the following conditions hold:

1) (X,D) is complete;
2) f is a t-contraction for some $t \in (0,1)$;
3) there exists $x_0 \in X$ such that $\delta(D,f,x_0)<\infty$.

Then $\{f^n(x_0)\}$ converges to $\omega \in X$, a fixed point of f. Moreover, if $\omega' \in X$ is another fixed pointiff such that $D(\omega,\omega')<\infty$, then $\omega=\omega'$.

D. Proof

Let $n \in N$ $(n \geq 1)$.
Since f is a k-contraction, for all $i,j \in N$,
we have
$D(f^{n+i}(x_0),f^{n+j}(x_0)) \leq k$
$D(f^{n-1+i}(x_0),f^{n-1+j}(x_0))$,
which implies that
$\delta(D,f,f^n(x_0)) \leq t\delta(D,f,f^{n-1}(x_0))$.
Then, for every $n \in N$,
we have
$\delta(D,f,f^n(x_0)) \leq t^n\delta(D,f,x_0)$.
Using the above inequality, for every $n,m \in N$, we have
$D(f^n(x_0),f^{n+m}(x_0)) \leq \delta$
$(D,f,f^n(x_0)) \leq t^n\delta(D,f,x_0)$.
Since $\delta(D,f,x_0)<\infty$ and $t \in (0,1)$, we obtain
$\lim_{n,m\to\infty} D(f^n(x_0),f^{n+m}(x_0))=0$,
which implies that $\{f^n(x_0)\}$ is a D-Cauchy sequence.
Since (X,D) is D-complete, there exists some $\omega \in X$ such that $\{f^n(x_0)\}$ is D-convergent to ω.
On the other hand, since f is a k-contraction, for all $n \in N$, we have
$D(f^{n+1}(x_0),f(\omega)) \leq tD(f^n(x_0),\omega)$.
Letting $n\to\infty$ in the above inequality,
we get

$$\lim_{n\to\infty} D(f^{n+1}(x_0),f(\omega))=0.$$

Then $\{f^n(x_0)\}$ is DD-convergent to $f(\omega)$.
By the uniqueness of the limit (see Proposition 2.4),
we get $\omega=f(\omega)$,
that is, ω is a fixed point of f.
Now, suppose that $\omega' \in X$ is a fixed point of f such that $D(\omega,\omega')<\infty$. Since f is a k-contraction, we have

$$D(\omega,\omega')=D(f(\omega),f(\omega')) \leq kD(\omega,\omega'),$$

which implies by the property (D1) that $\omega=\omega'$.
Observe that we can replace condition (iii) in Theorem 3.3 by
(iii)′: there exists $x_0\in X$ such that $\sup\{D(x_0,f^r(x_0)):r\in N\}<\infty$.
In fact, since *f* is a *k*-contraction,
we obtain easily that $\delta(D,f,x_0)\leq\sup\{D(x_0,f^r(x_0)):r\in N\}$.
So condition (iii)′ implies condition (iii).
The following result (see Kirk and Shahzad [6]) is an immediate consequence of Proposition 2.8 and Theorem 3.3.

1) *Corollary 3.4*

Let (X,d) be a complete b-metric space and $f:X\to X$ be a mapping. Suppose that for some $t\in(0,1)$, we have
$d(f(x),f(y))\leq td(x,y)$, for every $(x,y)\in X\times X$.
If there exists $x_0\in X$ such that $\sup\{d(f^i(x_0),f^j(x_0)):i,j\in N\}<\infty$,
then the sequence $\{f^n(x_0)\}$ converges to a fixed point of f. Moreover, f has one and only one fixed point.
Note that in [1], there is a better result than this given by Corollary 3.4.
The next result is an immediate consequence of Proposition 2.10 and Theorem 3.3.

2) *Corollary 3.5*

Let (X,d) be a complete dislocated metric space and $f:X\to X$ be a mapping. Suppose that for some $t\in(0,1)$, we have
$d(f(x),f(y))\leq td(x,y)$, for every $(x,y)\in X\times X$.
If there exists $x_0\in X$ such that
$\sup\{d(f^i(x_0),f^j(x_0)):i,j\in N\}<\infty$,
then the sequence $\{f^n(x_0)\}$ converges to a fixed point of f. Moreover, f has one and only one fixed point.
The following result is an immediate consequence of Proposition 2.15, Proposition 2.16, and Theorem 3.3.

3) *Corollary 3.6*

Let (X_q,q) be a complete modular space and $f:X\to X$ be a mapping. Suppose that for some $t\in(0,1)$, we have
$q(f(x)-f(y))\leq kq(x-y)$, for every $(x,y)\in X_q\times X_q$.
Suppose also that q satisfies the Fatou property. If there exists $x_0\in X_q$ such that $\sup\{q(f^i(x_0)-f^j(x_0)):i,j\in N\}<\infty$,
then the sequence $\{f^n(x_0)\}$ q-converges to some $\omega\in X_q$, a fixed point of f. Moreover, if $\omega'\in X_q$ is another fixed point of f such that $q(\omega-\omega')<\infty$, then $\omega=\omega'$.
Observe that in Corollary 3.6, no Δ_2-condition is supposed.

4 Ćirić's quasicontraction in a generalized metric space

In this section, we extend Ćirić's fixed point theorem for quasicontraction type mappings [21] in the setting of generalized metric spaces.
Let (X,D) be a generalized metric space and $f:X\to X$ be a mapping.

4) *Definition 4.1*

Let $k\in(0,1)$. We say that *f* is a *t*-quasicontraction if
$D(f(x),f(y))\leq t\max\{D(x,y),D(x,Tx),D(y,Ty),D(x,Ty),D(y,Tx)\}$,
for every $(x,y)\in X\times X$.

5) *Proposition 4.2*

Suppose that f is a k-quasicontraction for some $k\in(0,1)$. Then any fixed point $\omega\in X$ of f satisfies $D(\omega,\omega)<\infty\Longrightarrow D(\omega,\omega)=0$.

6) *Theorem 4.3*

Suppose that the following conditions hold:
1) (X,D) is complete;
2) f is a t-quasicontraction for some $t\in(0,1)$;
3) there exists $x_0\in X$ such that $\delta(D,f,x_0)<\infty$.

Then $\{f^n(x_0)\}$ converges to some $\omega\in X$. If $D(x_0,f(\omega))<\infty$ and $D(\omega,f(\omega))<\infty$, then ω is a fixed point off. Moreover, if $\omega'\in X$ is another fixed point of f such that $D(\omega,\omega')<\infty$ and $D(\omega',\omega')<\infty$, then $\omega=\omega'$.

E. Proof

Let $n\in N$ $(n\geq1)$.
Since *f* is a *t*-quasicontraction, for all $i,j\in N$,
we have
$D(f^{n+i}(x_0),f^{n+j}(x_0))\leq k\max\{D(f^{n-1+i}(x_0),f^{n-1+j}(x_0)),$
$D(f^{n-1+i}(x_0),f^{n+i}(x_0)),D(f^{n-1+i}(x_0),f^{n+j}(x_0)),$
$D(f^{n-1+j}(x_0),f^{n+j}(x_0)),D(f^{n-1+j}(x_0),f^{n+i}(x_0))\}$,
which implies that
$\delta(D,f,f^n(x_0))\leq t\delta(D,f,f^{n-1}(x_0))$.
Hence, for any $n\geq1$,
we have
$\delta(D,f,f^n(x_0))\leq t^n\delta(D,f,x_0)$.
Using the above inequality, for every $n,m\in N$,

we have
D(fn(x0),fn+m(x0))≤δ(D,f,fn(x0))≤knδ(D,f,x0).
Since δ(D,f,x0)<∞ and t∈(0,1),
we obtain limn,m→∞D(fn(x0),fn+m(x0))=0,
which implies that {fn(x0)} is a D-Cauchy sequence.
Since (X,D) is D-complete, there exists some ω∈X such that {fn(x0)} is D-convergent to *ω*.
Now, we suppose that D(x0,f(ω))<∞. Using the inequality

D(fn(x0),fn+m(x0))≤tnδ(D,f,x0), (4.1)

for every n,m∈N, by the property (D3), there exists some constant C>0 such that

D(ω,fn(x0))≤Clim supm→∞D(fn(x0),fn+m(x0))≤Ctnδ(D,f,x0), (4.2)

for every n∈N.
On the other hand, we have
D(f(x0),f(ω))≤tmax{D(x0,ω),D(x0,f(x0)),D(ω,f(ω)),D(f(x0),ω),D(x0,f(ω))}.
Using (4.1) and (4.2), we get
D(f(x0),f(ω))≤max{tCδ(D,f,x0),kδ(D,f,x0),kD(ω,f(ω)),kD(x0,f(ω))}.
Again, using the above inequality, we have
D(f2(x0),f(ω))≤max{t2Cδ(D,f,x0),k2δ(D,f,x0),kD(ω,f(ω)),k2D(x0,f(ω))}.
Continuing this process, by induction we get D(fn(x0),f(ω))≤max{tnCδ(D,f,x0),knδ(D,f,x0),kD(ω,f(ω)),knD(x0,f(ω))},
for every n≥1. Therefore, we have
lim supn→∞D(fn(x0),f(ω))≤tD(ω,f(ω)),
since D(x0,f(ω))<∞ and δ(D,f,x0)<∞. Using the property (D3), we get
D(f(ω),ω)≤lim supn→∞D(fn(x0),f(ω))≤tD(ω,f(ω)),
which implies that D(f(ω),ω)=0 since D(ω,f(ω))<∞ and k∈(0,1). Then *ω* is a fixed point of *f*. By Proposition 4.2, we have D(ω,ω)=0.
Finally, suppose that ω′∈X is another fixed point of *f* such that D(ω,ω′)<∞ and D(ω′,ω′)<∞. By Proposition 4.2, we have D(ω′,ω′)=0. Since *f* is a *t*-quasicontraction, we get
D(ω,ω′)=D(f(ω),f(ω)′)≤kD(ω,ω′),
which implies that ω=ω′.

As in the previous section, from Theorem 4.3, we can obtain fixed point results for Ćirić's quasicontraction type mappings in various spaces including standard metric spaces, *b*-metric spaces, dislocated metric spaces, and modular spaces.

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