# A Stability Analysis for Linear Delay-Differential Systems

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## Abstract

In this paper, we examine the problem of the stability analysis for linear delay-differential systems. Using Lyapunov method, we present sufficient conditions for the stability of the systems in terms of linear matrix inequality (LMI) Based on the Lyapunov–Krasovskii functional techniques which can be easily solved by using YALMIP Tool box. Numerical examples are given to illustrate our results.

**Keywords: Linear Delay-Differential Systems, Lyapunov Function, Linear Matrix Inequality (LMI), Time Delay, Delay Dependence**

## I. INTRODUCTION

Over the decades, the stability analysis of various delay-differential systems such as Neural Network systems, robust stability analysis, $H_2$&$H_\infty$ system of equations air-craft stabilization, control-systems stability problems, microwave oscillator, and many more engineering problems are done by many Researchers in various methodologies.

In this Linear Matrix Inequality tool playing a very important role, Utilizing of Lyapunov function generate a LMI which should be satisfied necessary and sufficient conditions for the stability. This can be applied for various systems of equations, it is one of the broad way of analyzing stability problems. Here we utilize the sufficient conditions for asymptotic stability from the Ju.H (2000) and applied for various system of equation , it shows in our illustrative examples.

Cao and He (2004) and park (2002) shows that system of equations are globally stable in there articles, M.Liu (2006) shows that system of equations are exponentially stable in his paper. Here we using Ju.H (2000) conditions for asymptotic stability which shows as a Theorem 1 in our paper and applied to Cao and He (2004) and park (2002) problems and shows that it is initially a asymptotically stable with good stable value. And reduced Theorem 1 with C constant Matrix considered as zero which shows in Corollary 1 and Applied to M.Liu (2006) shown that asymptotically stable with good stability solutions.

## II. MAIN RESULT

Consider a Linear delay-differential system of the form

$$\dot{x}(t) = Ax(t) + Bx(t-h) + C\dot{x}(t-h) \qquad (1)$$

With the initial condition function

$$x(t_0+\theta) = \emptyset(\theta), \text{for all } \theta \in (-h, 0) \qquad (2)$$

$x(t) \in R^n$ is the state vector, $A, B$ and $C \in R^{n*n}$ are constant matrices, h is a positive constant time-delay$\emptyset(.)$. is the given continuously differentiable function on $(h, 0)$and the system matrix A is assumed to be a Hurwitz matrix. The system given in (1) often appears in the theory of automatic control or population dynamics. First, we establish a delay-independent criterion, for the asymptotic stability of the neutral delay-differential system (1) using Lyapunov method in terms of LMI.

### *A. Theorem 1*

System (1) is asymptotically stable for all $h \geq 0$, if there exist positive definite Matrices $P > 0$ and $R > 0$ satisfying the following LMI:

$$\begin{bmatrix} A^TP + PA + R + A^TA & PB + A^TB & PC + A^TC \\ B^TP + B^TA & B^TB - R & B^TC \\ C^TP + C^TA & C^TB & C^TC - I \end{bmatrix} < 0 \qquad (3)$$

### *B. Proof*

Let the Lyapunov functional candidate be

$$V = x^T(t)Px(t) + v_1 + v_2 \qquad (4)$$

Where

$$v_1 = \int_{-h}^{0} \dot{x}^T(t+s)\dot{x}(t+s)ds \qquad (5)$$

$$v_2 = \int_{-h}^{0} x^T(t+s)Rx(t+s)ds \qquad (6)$$

The time derivative of V along the solution of (1) is given by

$$\dot{V} = x^T(A^TP + PA)x + 2x^TPBx_h + 2x^TPC\dot{x}_h + \dot{v_1} + \dot{v_2} \quad (7)$$

Where $x, x_h, \dot{x}_h$ denote $x(t), x(t-h), \dot{x}(t-h)$ respectively. From 5 & 6 we obtain

$$\dot{v_1} = \dot{x}^T\dot{x} - \dot{x}_h{}^T\dot{x}_h$$

$$= x^TA^TAx + x_h^TB^TBx_h + \dot{x}_h^TC^TC\dot{x}_h - \dot{x}_h^T\dot{x}_h + 2x^TA^TBx_h + 2x^TA^TC\dot{x}_h + 2\dot{x}_h^TB^TC\dot{x}_h \quad (8)$$

$$\dot{v_2} = x^TRx - x_h{}^TRx_h. \quad (9)$$

Substituting 8 & 9 into 7 we have

$$\dot{V} = \begin{bmatrix} x(t) \\ x(t-h) \\ \dot{x}(t-h) \end{bmatrix}^T \begin{bmatrix} A^TP + PA + R + A^TA & PB + A^TB & PC + A^TC \\ B^TP + B^TA & B^TB - R & B^TC \\ C^TP + C^TA & C^TB & C^TC - I \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-h) \\ \dot{x}(t-h) \end{bmatrix} \quad (10)$$

### *C. Corollary 1*

System (1) is asymptotically stable for all$h \geq 0$, if there exist positive definite Matrices $P > 0$ and $R > 0$ by considering C unknown Matrix as zero satisfying the following LMI:

$$\begin{bmatrix} A^TP + PA + R + A^TA & PB + A^TB & PC + A^TC \\ B^TP + B^TA & B^TB - R & 0 \\ 0 & 0 & -I \end{bmatrix} < 0 \quad (11)$$

#### *1) Proof*

Let the Lyapunov functional candidate be

$$V = v_1 + v_2 + v_3 \quad (12)$$

Where

$$v_1 = x^T(t)Px(t)$$

$$v_1 = \int_{-h}^{0} \dot{x}^T(t+s)\dot{x}(t+s)ds \quad (13)$$

$$v_3 = \int_{-h}^{0} x^T(t+s)Rx(t+s)ds$$

Taking derivative of the Lyapunov-Krasovskii functional and rearranging the terms results equation (10)

$$\dot{V} = \begin{bmatrix} x(t) \\ x(t-h) \\ \dot{x}(t-h) \end{bmatrix}^T \begin{bmatrix} A^TP + PA + R + A^TA & PB + A^TB & PC + A^TC \\ B^TP + B^TA & B^TB - R & B^TC \\ C^TP + C^TA & C^TB & C^TC - I \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-h) \\ \dot{x}(t-h) \end{bmatrix} \quad (14)$$

The Unknown Matrix considering as zero we get

$$\begin{bmatrix} A^TP + PA + R + A^TA & PB + A^TB & PC + A^TC \\ B^TP + B^TA & B^TB - R & 0 \\ 0 & 0 & -I \end{bmatrix} \quad (15)$$

### *D. Remark*

Now days many researchers are working on Stability with time delay system of equations using Lyapunov function. This proves gives the alternate solution for time delay system. Even this Theorem exist in (Ju.H -2000), In order to show the superiority of the results than the existing; we continued the work and reduced the theorem by considering with unknown matrix as zero shows in Corollary 1.

## III. NUMERICAL EXAMPLES

To illustrate the usefulness of the proposed approach, we present the following three simulation examples are given

### *A. Example 1: Consider the following system (Cao and He 2004)*

$$\dot{x}(t) = Ax(t) + Bx(t-h) + C\dot{x}(t-h) \quad (16)$$

Where $A = \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 0.2 & 0.5 \\ -0.3 & -0.4 \end{bmatrix}, C = \begin{bmatrix} 0.3 & 0.1 \\ -0.2 & 0.1 \end{bmatrix}$

Example 1 is Example 4.2 in (Cao and He 2004), whose approach is to judge the global asymptotical stability of system equation. However, according to Theorem 1, solving the linear delay Differential system of equation by invoking the LMI solver of YALMIP, we obtain the solutions as maximum allowable delay bound of time varying delay is 1.193577 with P and R values correspondingly

$$P = \begin{bmatrix} 4.3921 & 3.7271 \\ 3.7271 & 8.5449 \end{bmatrix}, R = \begin{bmatrix} 5.1640 & 1.9915 \\ 1.9915 & 4.0202 \end{bmatrix}$$

### *B. Example 2: Consider the following system (park 2002)*

$$\dot{x}(t) = Ax(t) + Bx(t-\tau) + C\dot{x}(t-h) \quad (17)$$

Where

$$A = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 \\ 0.9 & 0 \end{bmatrix}, C = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.4 \end{bmatrix}$$

Example 2 is in (park 2002), whose approach is to judge the global asymptotical stability of system equation. However, by applying Theorem 1, solving the Differential equation by invoking the LMI solver of YALMIP, we get the solution satisfies with the minimum allowable Lower bound -0.029172 with P and R values correspondingly

$$P = \begin{bmatrix} 1.8625 & 0 \\ 0 & 1.6980 \end{bmatrix}, R = \begin{bmatrix} 1.9272 & 0 \\ 0 & 1.6393 \end{bmatrix}$$

### *C. Example 3: Consider the following system (M.Liu 2006)*

$$\dot{x}(t) = Ax(t) + A_d x(t-h) \qquad (18)$$

Where

$$A = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}, A_d = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}$$

Example 3 is in (M.Liu 2006), whose approach is to judge the exponentially asymptotical stability of system equation, Here by using Corollary.1 solving the Differential equation by invoking the LMI solver of YALMIP, we obtain the solutions as maximum allowable delay bound of time varying delay 0.790096 with P and R values correspondingly

$$P = \begin{bmatrix} 4.4197 & 0 \\ 0 & 0 \end{bmatrix}, \quad R = \begin{bmatrix} 6.2231 & 0 \\ 0 & 0 \end{bmatrix}$$

## IV. CONCLUSION

Here we study about the sufficient conditions for Asymptotic stability using the existing Theorem 1 in Ju.H (2000) and reduced the theorem by considering unknown matrix as zero and shows in corollary 1 .The results of this paper indeed give us one more alternative for the stability examination of Linear delay-differential systems. The results were obtained in the literature with the help of YALMIP solver. In these our result has shown systems of equations are asymptotically stable with good conservative results.

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