A Study on Graphs of Groups on Surfaces

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Abstract

This paper discusses the definitions of Graphs, Groups and surfaces and some of their relations. The example for application of groups of graphs and surfaces in the form of change ringing are briefly discussed here.
Keywords: Graphs, Order of a graph, Automorphism Group of a Graph, Permutation Groups, Surfaces and Change Ringing

I. Introduction

Felix Klein (1872) Erlangen programme provides a broad frame work to understand the Topology of the space in terms of Geometry and that become a standard approach to the problems relating Geometry and Topology. In understanding the topology and geometry on low dimensional manifolds, opened up leading to some exciting research. Carl Fredrick Gauss initiated in 1828 the consideration of the curved surfaces which gave the first result which constituted the fundamental work of curved surfaces. He tried to connect a global result on curvature in terms of local information. Later in 1848 E. Bonnet generalized this and what we see is a central result that connects this global information namely the Gauss Bonnet theorem. Further, we see this catching up in its most generating form where the topological geometry invariants (global) were presented (S. S. Chern, Andraweal etc.) taking credit. B. Reimann student of Gauss was trying to understand the Geometry of surfaces with this idea from Complex Analysis which later came to be known as Reimann surfaces. Basically he was looking for a simple presentation of a surface.

On the other hand Euler though Eulerian graphs gives a clear picture (graphic account of the topological characterization of a surface and his famous Eulers formula (Euler's characteristics associated with the surface), Euler's characteristics. These historical facts are the comfort zones for us. Interestingly, Felix Klein (1872) Erlangen programme provides a broad frame work to understand the Topology of the space in terms of Geometry and that become a standard approach to the problems relating Geometry and Topology.

Here we put forth some ideas of Graphs, Groups and Surfaces.

II. Graphs of Groups on Surfaces

Graphs: For those who have heard these terms, there is no need to tell that what a graph or group is for that matter. But for the knowledge of those who are uninitiated requires a brief description about them. This is what we do. One regards a graph as a discrete structure. By a structure we mean some basic relations to the entity which has an underlying set whose elements are the elements of the entity. Thus a graph will have an underlying set X and the basic relation that we talk will come from various Cartesian powers of X and in turn help us to know the definable sets. In case of a graph we have two definable sets, namely the vertex set V(X) and the edge set E(X).Thus a graph X is a pair consisting of vertex set and edge set {V(X), E(X)}. Points of V(X) are called the vertices of the graph. Relation is a subset of a Cartesian set. V(X) = {(u,v), u,v are the elements of X called vertices} (u,v) ϵ X.

III. Order of the Graph

The edge set E(X) is defined as that set which whenever two vertices are picked up, if there is incidence between u and v then we say that this incidence forms an edge between u and v. Thus E(X) = {(u, v)}, u v ϵ X and u and v are incident. This incidence relation will give an edge set. V(X), E(X) gives the size of the sets. This V(X) also constitutes the order of the graph. Observe that the edge set can be empty. This is the case when V(X) = singleton. Therefore for E(X) = φ to be nonempty we require V(X) = at least two. A nontrivial situation is E(X) $\neq \phi$.

A. Assumption

All our graphs are planar means they can be embeddable in a plane. If we have a plane which is a two dimensional surface on which these graphs are embedded and is noncompact. So, what if the two dimensional surface is compact? (2 dimension). We dispense this for the case of simple closed surfaces. For example the sphere S^2 is a simple closed compact surface. Another example is S^2 with k- handles.

IV. GROUPS

Like a graph a group is also a mathematical object that describes the algebraic nature of the entity. It can be discrete or continuous (unlike the graph). Group structure with basic relation has an underlying set called G and a binary operation on G that is for basic relation into itself. $GxG \rightarrow G$ * o, aob → ab which satisfies

1) o is associative and possesses a distinguished element 'e' with the property aoe = eoa for a $\in$ G and for each a $\in$ G the distinguished element y_a with the property $y_a oa = eoy_a = e$ and y_a constitutes the inverse of a simply denoted by a^{-1}. Rest all are additional properties.

Abelean: If a, b $\in$ X and aob = boa i.e. the commutative property is satisfied then we say that the group is abelean. Basically groups are for symmetry.

Each group can be defined in terms og generators and relations and corresponding to such a presentation there is a unique graph called the Cayley color graph of the presentation. A drawing of this graph gives a picture of the group.

V. GROUPS ARISING FROM GRAPHS

A familiar approach to relate a graph to a group is by means of permutations which are automorphs (one-one, onto, bijective) maps from a set {1,2,….. N} onto itself. These are permutations defined on a finite set. If f: {1,2,….. N} → {1,2,….. N} is a permutation i.e i → f(i) for each i, $1 \leq i \leq N$ the image f(i) of i under f could be any j for j to have N choices.

Thus, we can write permutation of f as f : (1,2,….. N). Observe that f and g are two permutations then their composition is well defined. gof will again be a permutation. If S_N = {f/f : (1,2,….. N) →(1,2,….. N)} is a permutation then S_N = N! <S_N , 0>. S_N is a binary operation and a group. We observe that S_N is non- commutative. It is a finite group with order N. Permutation groups are non-commutative. The set here namely (1,2,… N) is called the object set(or the underlying set for the permutation group) if SN denotes the permutation groups for the object set (1,2,….. N). Denote S_N by A and (1,2,….. N) by X. Then A is the order of the group and X is the degree. X associated with X is its group A call it as a permutation map. Here our underlying set is the group.

VI. AUTOMORPHISM GROUP OF A GRAPH

We define Automorphism of a graph as a permutation of V(G) preserving adjacency.

A. Definition:

An Automorphism of a graph is a graph isomorphism with itself, i.e., a mapping from the vertices of the given graph G back to vertices of G such that the resulting graph is isomorphic with G. The set of Automorphism defines a permutation group known as the graph's Automorphism group

B. Definition:

A one-to-one mapping from a finite set onto itself is called a permutation. A permutation group is a group whose elements are all permutations acting on the same finite set called the object set. The group operation is composition of mappings.

C. Definition:

An Automorphism of g, which is an isomorphism of G with itself. The set of all Automorphisms of G forms a permutation group. Aut(G) acting on the object set V(G). Aut(G) is called the Automorphism group of G.
Note: An Automorphism of G, which is a permutation of V(G) also induces a permutation of E(G).

VII. PERMUTATION GROUPS

A. Definition:

A one-to one mapping from a finite set onto itself is called a permutation. A permutation group is a group whose elements are all permutations acting on the same finite set, called the object set. If X is the object set and A the permutation group, then A is the order of the group and X is the degree.

A permutation P partitions its object set by the equivalence relation x y if and only if $P^k(x)=y$ for some integer k. The equivalence classes are called the orbits of X under the action of P. If there is just one orbit in the action of X on A, then A is said to be transitive on X. If A =X and if A is transition on X, then A is said to be a regular permutation group and X is the degree.

B. Operations on Permutations Groups:

Let A and B be permutation groups acting on object sets X and Y respectively. We define three binary operations on these permutation groups as follows:

1) The sum AxB acts on the disjoint union XUY: A+B = {a+b/a $\in$ A, b $\in$ B}, and
   (a+b) (z) = az, if z $\in$ X
   = bz, if z $\in$ Y.
2) The product AxB, acts on XxY; AxB = {axb/a $\in$ A, b $\in$ B}, (axb) (x, y) = (ax, by).
3) The composition, A[B], acts on XxY as follows for each a $\in$ A and any sequence $b_1, b_2 \ldots\ldots b_d$ in b, there is a unique permutation in A[B], written (a, $b_1, b_2 \ldots\ldots b_d$) and (a, $b_1, b_2 \ldots\ldots b_d$)($x_i$, y_j) = (ax_i, by_j).

Thus A[B] ={(a , f) / a $\in$ A, f : X$\rightarrow$B }, (a,f) (x,y) =(ax, b_xy) where f(x) =b_x.

VIII. A SURFACE

Intuitively we look at the set R, the reals, R^2 the Cartesian product on R taken two times. As topological spaces R admits a standard topology in which the open sets are intervals and we get a real line once the notion of connectedness is characterized on R. Product topology induced on R^2 gives us a 2-dimensional surface. An arbitrary surface can be thought over as an homeomorphic copy of R^2.

Otherwise we have the familiar notion of defining a plane in R^3 by the set of all (x, y, z) $\in$ R^3.

Triangulability or triangulation was the key to achieve this objective for surfaces (all our surfaces are simple closed oriented surfaces). It could be a sphere. How to triangulate a sphere X = S^2, X = T^2.

A surface will be a closed orientable 2-manifold and any figure may be considered as a topological space in a Euclidean space. Here the subspace topology is the induced topology coming from the standard Euclidean metric in R^3.

To get an idea of a surface in these technical terms which we now describe by an open disc D we mean D = {(x,y) $\in$ R^2 / x^2 + y^2 <1}.

A. Definition:

A 2- manifold is a connected topological space in which every point has a neighborhood homeomorphic to the open unit disk.
If X is connected topological space and x $\in$ X is a point in X there is a neighborhood in the point Nx. Then this N_x is homeomorphic to D.
A manifold is a generalization of Euclidean space.
Likewise an n-manifold is a topological space in which every point x has a neighborhood which is homeomorphic to B_n = { x_1, x_2,….x_n / xi2 < 1}

B. Definition:

A subspace *M* of R^2 is bounded if there exists a natural number N such that $M \subset$ B(0, n) = {(x, y, z) $\in$ R^3/ $x^2 + y^2 + z^2 < x^2$}.

C. Definition:

Let $M \subset$ R^3 be a 2-manifold then M is said to be bounded and the boundary of M coincides with M.

IX. CAYLEY THEOREM

It shows that every group is isomorphic to a group of permutations. That is, every group is isomorphic to a subgroup of the symmetric group on some suitable set.

Cayley Theorem: Every group is isomorphic to a group of permutations. Hence proving f is a homomorphism.

X. CHANGE RINGING

A mathematical model for change ringing uses graphs and groups on surfaces. The first observation is that a change of n bells is a permutation of a degree and that, the symmetric group S_n is relevant on n bells. Secondly each legal transition can be described as a product of disjoint transpositions of adjacent numbers from {1, 2, ….n} – regarded as a line, not a circle, so hat 1, 2, ...n are not adjacent (for n $\geq$ 3).

The ancient and continuing art of change ringing, or campanology, is studied from mathematical point of view. An extent on n bells is regarded as a hamiltonian cycle in a Cayley color graph for the symmetric group S_n, often imbedded in an appropriate surface. Thus graphs, groups and surfaces combine to model something musical.

Bells can be made to follow one another in order, each ringing once before the first rings again. Ringing bells in a precise relationship to one another is the essence of change ringing. Rung in the order from the lightest, highest pitched bell to the heaviest, deepest pitched bell, they strike in a sequence known as ‘rounds’. To produce variations in the sound, bells are made to change places with adjacent bells in the row. In order to ring a different row with each pull of the rope, ringers have devised methods, orderly systems of changing pairs. In ringing a method the bells begin in rounds, ring changes according to the method, and return to rounds without repeating any row along the way. Ringers show methods and the bells in them using numbered pattern. To remember where they must strike their bell they learn a ‘blue line’ – which shows their bell’s path.

REFERENCES

[1] Arther T White, Graphs of Groups on Surfaces
[2] Abay-Asmerom, G., Graph Products and Covering Graph Embeddings, Phd Dissertaton, Western Michigan University, 1990.
[3] Anderson S, Graph Theory and Finite Combinatorics, Markham, Chicago, 1970,
[4] Biggs, N.,L., Algebraic Graph Theory, Secnodn Edition, Cambridge University Press, Cambridge, 1993.
[5] L Grossman and W Magnus, Groups and heir Graphs, Random house , New York, 1964.
[6] L.W Beinke and R.J.Wilson, selected Topics in Graph Theory, Academic Press, London,1978.
[7] F. Harary, Graph Theory, Addison Wesley, Reading, Mass,1969