Relation between Fuzzy Subgroups & Anti-Fuzzy Subgroups

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Abstract

In this paper, we discuss the definitions of union of Fuzzy subsets, the definitions of union of Anti-Fuzzy subsets, definition of fuzzy subgroup, definition of anti-fuzzy subgroup, and show the relation between fuzzy subgroups and an Anti-fuzzy subgroups and derive some theorems.

Keywords: Fuzzy Subsets, Anti-Fuzzy Subsets, Compact, Closed Sets, Compositions

I. INTRODUCTION

The concept of fuzzy sets was initiated by Zadeh. Then it has become a vigorous area of research in engineering, medical science, social science, graph theory etc. Since the paper fuzzy set theory has been considerably developed by zadeh himself and some researchers. The original concept of fuzzy sets was introduced as an extension of crisps (usual) sets, by enlarging the truth value set of “grade of members” from the two value set \{0,1\} to unit interval \{0,1\} of real numbers. The study of fuzzy group was started by Rosenfeld. It was extended by Roventa who have introduced the fuzzy groups operating on fuzzy sets.

II. PRELIMINARIES

In this section contain some definitions, examples and some results.

A. Concept of a Fuzzy Set

The concept of a fuzzy set is an extension of the concept of a crisp set. Just as a crisp set on a universal set U is defined by its characteristic function from U to \{0,1\}, a fuzzy set on a domain U is defined by its membership function from U to [0,1].

Let U be a non-empty set, to be called the Universal set (or) Universe of discourse or simply a domain.

Then, by a fuzzy set on U is meant a function A: U \rightarrow [0,1]. A is called the membership function; A(x) is called the membership grade of x in A. We also write

A= \{(x, A(x)) : x \in U\}.

1) Examples

Consider \(U=\{a, b, c, d\}\) and \(A: U \rightarrow 1\) defined by \(A(a)=0, A(b)=0.7, A(c)=0.4, \text{and } A(d)=1\). Then A is a fuzzy set can also be written as follows:

\[ A= \{(a, 0), (b,0.7),(c,0.4),(d,1)\}. \]

B. Relation between Fuzzy Sets

Let U be a domain and A, B be fuzzy sets on U.

Inclusion (or) Containment: A is said to be included (or) contained in B if and only if \(A(x) \leq B(x)\) for all x in U.

In symbols, we write, \(A \subseteq B\). We also say that A is a subset of B.

C. Definition

Let S be a set. A fuzzy subset A of S is a function A: S \rightarrow [0,1].

D. Definition of Union of Fuzzy sets:

The union of two fuzzy subsets \(\mu_1, \mu_2\) is defined by \((\mu_1 \cup \mu_2)(x) = \max\{\mu_1(x), \mu_2(x)\}\) for every x in U.

E. Definition of Fuzzy Union of the fuzzy sets \(\mu_1\) and \(\mu_2\)

Let \(\mu_1\) be a fuzzy subset of a set \(x_1\) and \(\mu_2\) be a fuzzy subset of a set \(x_2\), then the fuzzy union of the fuzzy sets \(\mu_1\) and \(\mu_2\) is defined as a function.
μ₁ ∪ μ₂ : x₁ ∪ x₂ → [0,1] given by

\[
(μ₁ ∪ μ₂)(x) = \begin{cases} 
\max(μ₁(x),μ₂(x)) & \text{if } x ∈ x₁ ∩ x₂, \\
μ₁(x) & \text{If } x ∈ x₁ & x ∉ x₂, \\
μ₂(x) & \text{If } x ∈ x₂ & x ∉ x₁.
\end{cases}
\]

### F. Definition of Union of Anti-Fuzzy Sets

The union of two fuzzy subsets A₁, A₂ is defined by (A₁ ∪ A₂)(x) = min{A₁(x), A₂(x)} for every x in U.

### G. Definition of Anti-Fuzzy union of the fuzzy sets A₁ and A₂

Let A₁ be a fuzzy subset of a set x₁ and A₂ be a fuzzy subset of a set x₂, then the anti-fuzzy union of the fuzzy sets A₁ and A₂ is defined as a function.

A₁ ∪ A₂ : x₁ ∪ x₂ → [0,1] given by

\[
( A₁ ∪ A₂)(x) = \begin{cases} 
\min(A₁(x), A₂(x)) & \text{if } x ∈ x₁ ∩ x₂, \\
A₁(x) & \text{If } x ∈ x₁ & x ∉ x₂, \\
A₂(x) & \text{If } x ∈ x₂ & x ∉ x₁.
\end{cases}
\]

### II. Definition of Fuzzy Subgroup

Let G be a group. A fuzzy subset μ of a group G is called a fuzzy subgroup of the group G if

1) \( μ(xy) ≥ \min \{μ(x),μ(y)\} \) for every \( x,y \in G \). And

2) \( μ(x^{-1}) = μ(x) \) for every \( x ∈ G \).

### I. Definition of Anti Fuzzy Subgroup

Let G be a group. A fuzzy subset μ of a group G is called an anti-fuzzy subgroup of the group G if

1) \( μ(xy) ≤ \max \{μ(x),μ(y)\} \) for every \( x,y \in G \) and

2) \( μ(x^{-1}) = μ(x) \) for every \( x ∈ G \).

From the definitions 2.8. And 2.9. We have min \( \{μ(x),μ(y)\} ≤ μ(xy) ≤ \max \{μ(x),μ(y)\} \) \tag{1}

### III. Theorems

#### A. Theorem 3.1

By condition 1, prove that μ is compact.

1) **Proof**

w.r.t

By Hein–Borel Theorem “The set is compact if and only if it is closed and bounded”.

It is enough to prove that μ is closed and bounded.

The equation 1 is compared with \( a ≤ x ≤ b \), it is closed.

Obviously, μ is closed.

By I 

\( μ(xy) ≤ \max \{μ(x),μ(y)\} = M \).

Obiously μ is bounded.

Thus μ is closed and bounded.

Hence μ is compact.

#### B. Theorem 3.2

A Closed subset of a compact set is compact.

1) **Proof**

Let \( μ₁ \) and \( μ₂ \) are subsets of a compact set, \( μ₁ ⊆ μ \) And \( μ₂ ⊆ μ \)

By I,
From 2 and 3, Thus $\mu_1$ and $\mu_2$ are closed and bounded.
By Theorem 1,
Hence $\mu_1$ and $\mu_2$ are compact.

C. Theorem 3.3

A Finite union of compact sets is compact.

1) Proof

Let $\mu_1$ and $\mu_2$ are compact subsets of a compact set ,
$$\mu_1 \subseteq \mu_1 \cup \mu_2 \subseteq \mu \quad \text{and} \quad \mu_2 \subseteq \mu_1 \cup \mu_2 \subseteq \mu .$$

by theorem 2.
Thus $\mu_1 \cup \mu_2$ is closed and bounded.
Thus $\mu_1 \cup \mu_2$ is compact.
Hence a Finite union of compact sets is compact.

D. Theorem 3.4

The Product of any collection of compact sets is compact.

1) Proof

Let $\mu_1$ and $\mu_2$ are compact subsets of a compact set ,
By above theorems,
$\mu_1 \circ \mu_2$ is also compact.

E. Theorem 3.5:

If $\mu_1 \circ \mu_2$ is a fuzzy subgroup of a group $G$ if and only if $\mu_1 \circ \mu_2 = \mu_2 \circ \mu_1$.

1) Proof

a) Necessary Part

Let $\mu_1 \circ \mu_2 = \mu_2 \circ \mu_1$
To prove:
$\mu_1 \circ \mu_2$ is a fuzzy subgroup of a group $G$.
Let $\mu_1$ and $\mu_2$ are fuzzy subgroup of a group $G$
$\mu_1 \circ \mu_2(x) = \mu_1(\mu_2(x))$, where $x, y \in G$
Since $\mu_2$ is a fuzzy subgroup of a group $G$.
by the definition of fuzzy subgroup ,
1) $\mu_2(xy) \geq \min \{\mu_2(x), \mu_2(y)\}$ for every $x, y \in G$.
$\mu_1 \circ \mu_2(xy) \geq \mu_1(\min \{\mu_2(x), \mu_2(y)\})$
$\geq \min \mu_1(\{\mu_2(x), \mu_2(y)\})$
$\mu_1 \circ \mu_2(xy) \geq \min \{\mu_1 \circ \mu_2(x), \mu_1 \circ \mu_2(y)\}$
$\mu_1 \circ \mu_2(x^{-1}) = \mu_1(\mu_2(x^{-1}))$ \hspace{1cm} (1)

Again by the definition of fuzzy subgroup,
$\mu_2(x^{-1}) = \mu_2(x)$ for every $x \in G$.
$\mu_1 \circ \mu_2(x^{-1}) = \mu_1(\mu_2(x))$
$\mu_1 \circ \mu_2(x^{-1}) = \mu_1 \circ \mu_2(x)$ \hspace{1cm} (2)

From 1 and 2,
$\mu_1 \circ \mu_2$ is a fuzzy subgroup of a group $G$.

b) Sufficient Part

Let $\mu_1 \circ \mu_2$ is a fuzzy subgroup of a group $G$.
To prove:
$\mu_1 \circ \mu_2 = \mu_2 \circ \mu_1$
by 1 ,
$\mu_1 \circ \mu_2(xy) \geq \mu_2(\min \{\mu_1(x), \mu_1(y)\})$

Since by the definition of fuzzy subgroup ,
2) $\mu_1(xy) \geq \min \{\mu_1(x), \mu_1(y)\}$ for every $x, y \in G$.

Similarly, we get,
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\[ \mu_1 \circ \mu_2 \subseteq \mu_2 \circ \mu_1 \]  \hspace{1cm} (4)

From 3 and 4,

\[ \mu_1 \circ \mu_2 = \mu_2 \circ \mu_1 . \]

Hence,

If \( \mu_1 \circ \mu_2 \) is a fuzzy subgroup of a group G if and only if \( \mu_1 \circ \mu_2 = \mu_2 \circ \mu_1 \).

Thus proved.

For an Anti-fuzzy subgroups, we use A instead of \( \mu \).

\section*{F. Theorem 3.5}

If \( A_1 \circ A_2 \) is an Anti- fuzzy subgroup of a group G if and only if \( A_1 \circ A_2 = A_2 \circ A_1 \).

\subsection*{1) Proof}

\begin{enumerate}
\item [a)] Necessary part

Let \( A_1 \circ A_2 = A_2 \circ A_1 \)

To prove:

\( A_1 \circ A_2 \) is an Anti-fuzzy subgroup of a group G.

Let \( A_1 \) and \( A_2 \) are an Anti-fuzzy subgroup of a group G.

\( A_1 \circ A_2 (xy) = A_1(A_2(xy)) \), where \( x, y \in G \)

Since \( A_2 \) is an Anti-fuzzy subgroup of a group G.

by the definition of an Anti-fuzzy subgroup,

3) \( A_2(xy) \leq \max \{ A_2(x), A_2(y) \} \) for every \( x, y \in G \).

\( A_1 \circ A_2 (xy) \leq A_1(\max \{ A_2(x), A_2(y) \}) \)

\( \leq \max A_1(\{ A_2(x), A_2(y) \}) \)

\[ A_1 \circ A_2 (xy) \leq \max \{ A_1 \circ A_2 (x), A_1 \circ A_2 (y) \} \]  \hspace{1cm} (1)

\( A_1 \circ A_2 (x^{-1}) = A_1(A_2(x^{-1})) \)

Again by the definition of an Anti-fuzzy subgroup,

\( A_2(x^{-1}) = A_2(x) \) for every \( x \in G \).

\[ A_1 \circ A_2 (x^{-1}) = A_1(A_2(x)) \]  \hspace{1cm} (2)

From 1 and 2,

\( A_1 \circ A_2 \) is an Anti-fuzzy subgroup of a group G.

\item [b)] Sufficient Part

Let \( A_1 \circ A_2 \) is an Anti-fuzzy subgroup of a group G.

To prove

\( A_1 \circ A_2 = A_2 \circ A_1 \)

by 1.

\( A_1 \circ A_2 (xy) \leq \max \{ A_1(x), A_1(y) \} \)

Since by the definition of an Anti-fuzzy subgroup,

4) \( A_1(xy) \leq \max \{ A_1(x), A_1(y) \} \) for every \( x, y \in G \).

\[ A_1 \circ A_2 (xy) \leq A_2(A_4(xy)) \]

\[ A_1 \circ A_2 (xy) \leq A_2 \circ A_1 (xy) \]

\[ A_2 \circ A_1 \subseteq A_1 \circ A_2 \]  \hspace{1cm} (3)

Similarly,

We get,

\( A_1 \circ A_2 \leq A_2 \circ A_1 \)  \hspace{1cm} (4)

From 3 and 4,

\[ A_1 \circ A_2 \leq A_2 \circ A_1 \]

Hence,

If \( A_1 \circ A_2 \) is a fuzzy subgroup of a group G if and only if \( A_1 \circ A_2 = A_2 \circ A_1 \).

Thus proved.

\section*{IV. CONCLUSION}

We derive the Relation between Fuzzy subgroups and Anti-Fuzzy subgroups and shows that the set is compact and their union is also compact, compact properties and shown the composition properties.

\section*{REFERENCES}
